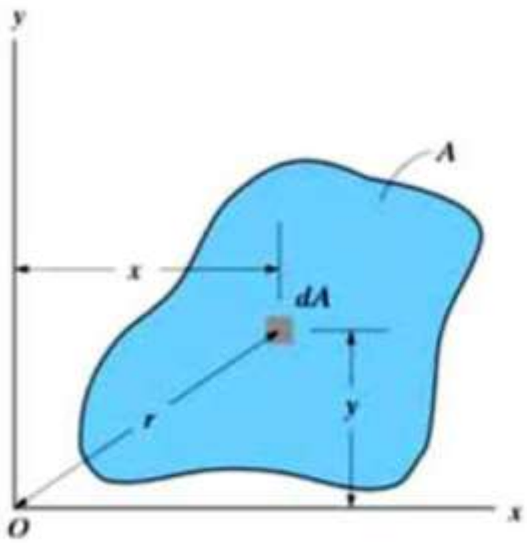
THE MOMENT OF INERTIAFOR AREA

The inertia is the resistance of any object to a change.
The moment of inertia is a measure of an object's resistance to changes its rotation.
The moment of inertia for an area is important property in analysis and design of structural members.
The centroid represents the moment of area ($\int xdA$), while the moment of area represents the second moment of area ($\int_A x^2 dA$).
Consider the figure:
The moment of inertia of the area about x & y axes are:

$$I_x = \int_A y^2 \, dA$$
$$I_y = \int_A x^2 \, dA$$



The polar moment of inertia J_o is:

$$J_O = \int_A r^2 \, dA = I_x + I_y$$

The moment of inertia is alawys positive (product of distance squered and area), and the units are length raised to the fourth power e.g. m^4, mm^4,….

Radius of Gyration of an Area:

The redius of gyration is often used in design of columns. The formulas are:

$$k_x = \sqrt{\frac{I_x}{A}}$$

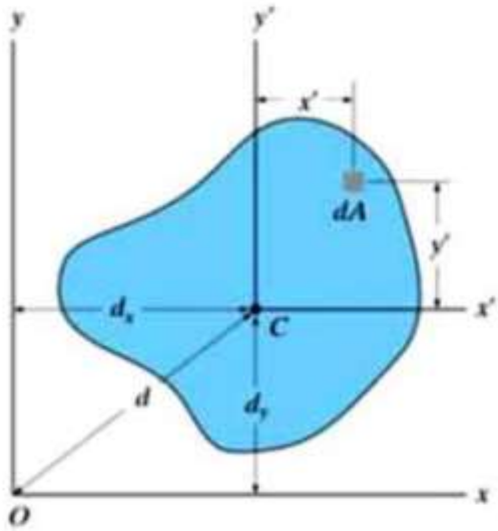
$$k_y = \sqrt{\frac{I_y}{A}}$$

$$k_O = \sqrt{\frac{J_O}{A}}$$

Parallel Axis Theorem for an Area:

This theorem is used to find the moment of inertia about an axis parallel to the axis passing through the centroid.

This theorem says:



$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$I_y = \bar{I}_{y'} + Ad_x^2$$

$$J_O = \bar{J}_C + Ad^2$$

Moment of Inertia for Composite Areas:

The following procedures provides a method for determing the moment of inertia of a composite areas about a reference axis.

1. Divide the area into its composite parts and indicate the perpendicular distance from the centroid of each part to the reference axis.

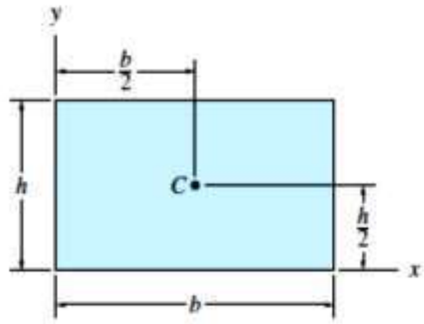
2. Find the moment of inertia of each part about its centroidal axis (use the table of moment of inertia). If the centroidal axis does not coincide with the reference axis; use the parallel axis theorem $I = \bar{I} + Ad^2$.

3. The moment of inertia of the total area about the reference axis is determing by summing the results of parts.

The table below shows the moment of inertia for more common shapes :

Moments of Inertia of Areas and Polar Moments of Inertia

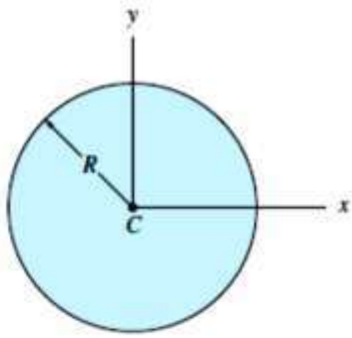
Rectangle



$\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_y = \frac{b^3h}{12}$ $\bar{I}_{xy} = 0$

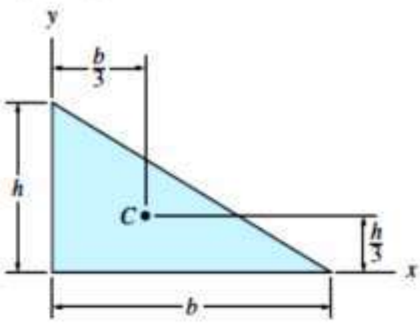
$I_x = \frac{bh^3}{3}$ $I_y = \frac{b^3h}{3}$ $I_{xy} = \frac{b^2h^2}{4}$

Circle



$I_x = I_y = \frac{\pi R^4}{4}$ $I_{xy} = 0$

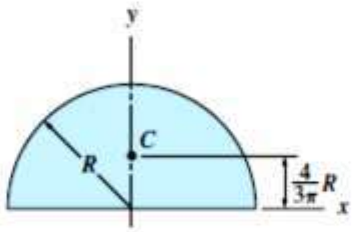
Right triangle



$\bar{I}_x = \frac{bh^3}{36}$ $\bar{I}_y = \frac{b^3h}{36}$ $\bar{I}_{xy} = -\frac{b^2h^2}{72}$

$I_x = \frac{bh^3}{12}$ $I_y = \frac{b^3h}{12}$ $I_{xy} = \frac{b^2h^2}{24}$

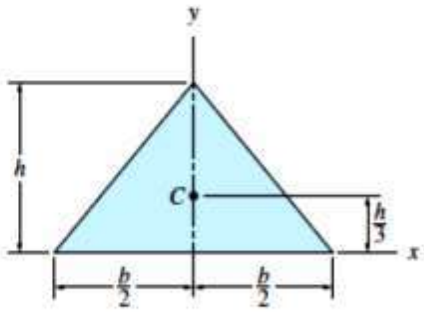
Semicircle



$\bar{I}_x = 0.1098R^4$ $\bar{I}_{xy} = 0$

$I_x = I_y = \frac{\pi R^4}{8}$ $I_{xy} = 0$

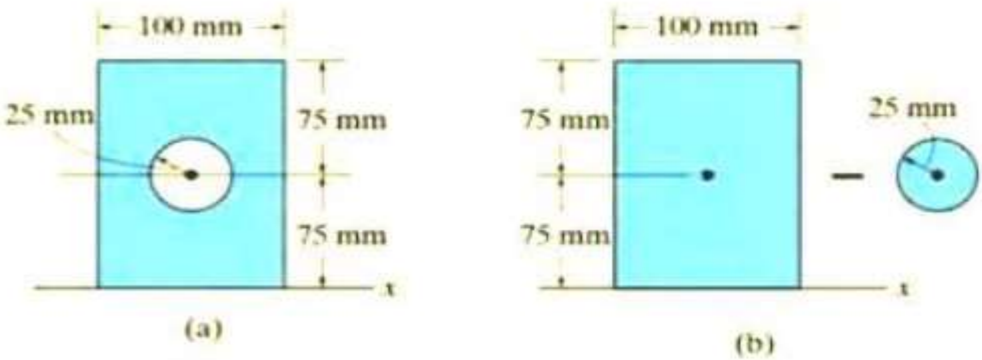
Isosceles triangle



$\bar{I}_x = \frac{bh^3}{36}$ $\bar{I}_y = \frac{b^3h}{48}$ $\bar{I}_{xy} = 0$

$I_x = \frac{bh^3}{12}$ $I_{xy} = 0$

Ex.1: Determine the moment of inertia of the area shown about the x axis.



Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\ \text{mm}^4$$

Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\ \text{mm}^4$$

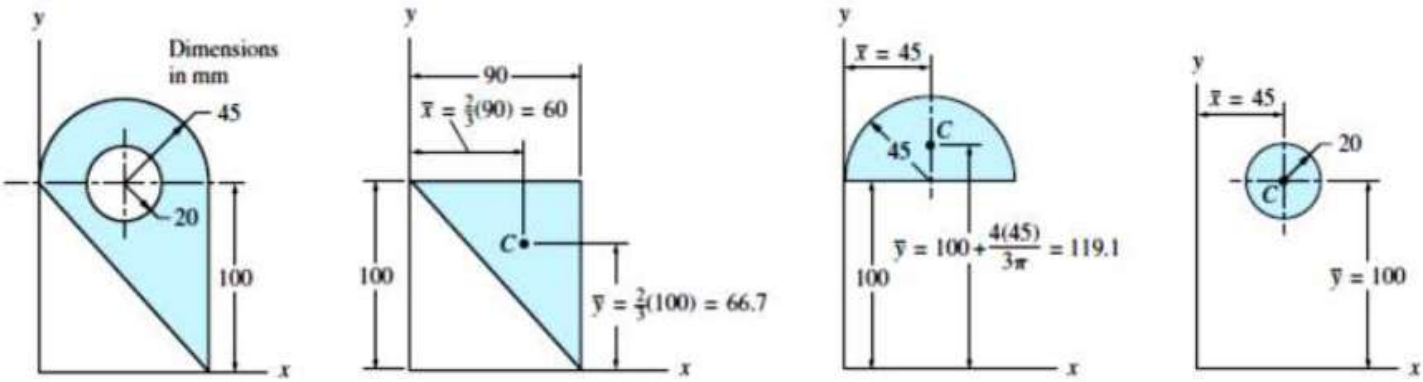
Summation. The moment of inertia for the area is therefore

$$I_x = -11.4(10^6) + 112.5(10^6)$$

$$= 101(10^6)\ \text{mm}^4$$

<u>**Ex.4:**</u> For the area shown in Fig. (a),
x- and y-axes.

<u>**Sol.:**</u>



Triangle

$$A = \frac{bh}{2} = \frac{90(100)}{2} = 4500 \text{ mm}^2$$

$$\bar{I}_x = \frac{bh^3}{36} = \frac{90(100)^3}{36} = 2.50 \times 10^6 \text{ mm}^4$$

$$I_x = \bar{I}_x + A\bar{y}^2 = (2.50 \times 10^6) + (4500)(66.7)^2 = 22.52 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = \frac{hb^3}{36} = \frac{100(90)^3}{36} = 2.025 \times 10^6 \text{ mm}^4$$

$$I_y = \bar{I}_y + A\bar{x}^2 = (2.025 \times 10^6) + (4500)(60)^2 = 18.23 \times 10^6 \text{ mm}^4$$

Semicircle

$$A = \frac{\pi R^2}{2} = \frac{\pi(45)^2}{2} = 3181 \text{ mm}^2$$

$$\bar{I}_x = 0.1098R^4 = 0.1098(45)^4 = 0.450 \times 10^6 \text{ mm}^4$$

$$I_x = \bar{I}_x + A\bar{y}^2 = (0.450 \times 10^6) + (3181)(119.1)^2 = 45.57 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = \frac{\pi R^4}{8} = \frac{\pi(45)^4}{8} = 1.61 \times 10^6 \text{ mm}^4$$

$$I_y = \bar{I}_y + A\bar{x}^2 = (1.61 \times 10^6) + (3181)(45)^2 = 8.05 \times 10^6 \text{ mm}^4$$

Circle

$$A = \pi R^2 = \pi(20)^2 = 1257 \text{ mm}^2$$

$$\bar{I}_x = \frac{\pi R^4}{4} = \frac{\pi(20)^4}{4} = 0.1257 \times 10^6 \text{ mm}^4$$

$$I_x = \bar{I}_x + A\bar{y}^2 = (0.1257 \times 10^6) + (1257)(100)^2 = 12.70 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = \frac{\pi R^4}{4} = \frac{\pi(20)^4}{4} = 0.1257 \times 10^6 \text{ mm}^4$$

$$I_y = \bar{I}_y + A\bar{x}^2 = (0.1257 \times 10^6) + (1257)(45)^2 = 2.67 \times 10^6 \text{ mm}^4$$

Composite Area

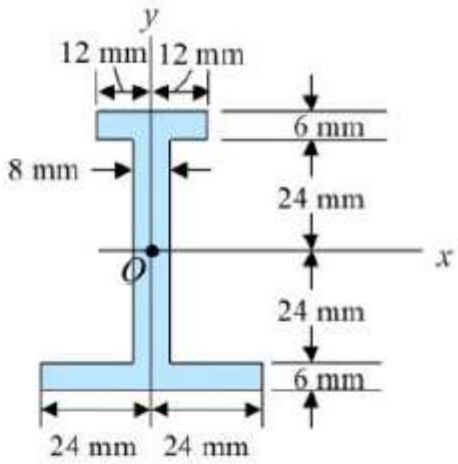
To determine the properties for the composite area, we superimpose the foregoing results (taking care to subtract the quantities for the circle) and obtain

$$A = \Sigma A = 4500 + 3181 - 1257 = 6424 \text{ mm}^2$$

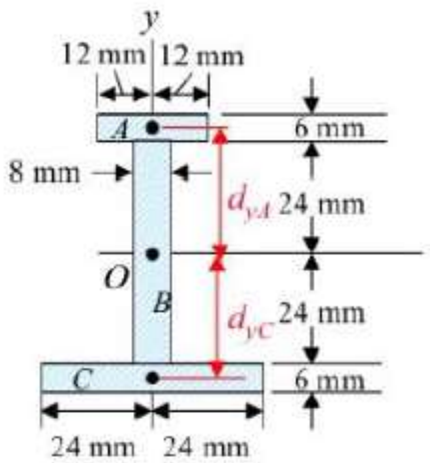
$$I_x = \Sigma I_x = (22.52 + 45.57 - 12.70) \times 10^6 = 55.39 \times 10^6 \text{ mm}^4$$

$$I_y = \Sigma I_y = (18.23 + 8.05 - 2.67) \times 10^6 = 23.61 \times 10^6 \text{ mm}^4$$

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.



SOLUTION



$$I_x = (\bar{I}_x + A d_y^{\,2})_A + (\bar{I}_x + \cancel{A d_y^{\,2}}^{\,0})_B + (\bar{I}_x + A d_y^{\,2})_C$$

$$= [\frac{1}{12}(24)(6)^3 + (24\times 6)(27)^2]_A$$

$$+ [\frac{1}{12}(8)(48)^3 + 0]_B$$

$$+ [\frac{1}{12}(48)(6)^3 + (48\times 6)(27)^2]_C$$

$$I_x = 390\text{x}10^3 \text{ mm}^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390\times 10^3}{[(24\times 6)+(8\times 48)+(48\times 6)]}} = 21.9\, mm$$

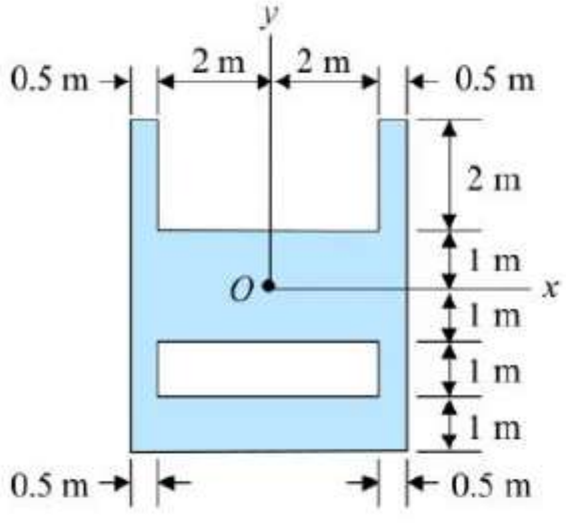
$$I_y = (\bar{I}_y + \cancel{A d_x^{\,2}}^{\,0})_A + (\bar{I}_y + \cancel{A d_x^{\,2}}^{\,0})_B + (\bar{I}_y + \cancel{A d_x^{\,2}}^{\,0})_C$$

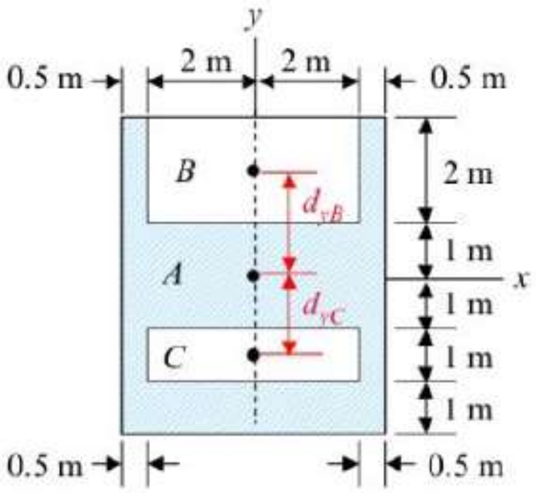
$$= [\frac{1}{12}(6)(24)^3]_A + [\frac{1}{12}(48)(8)^3]_B + [\frac{1}{12}(6)(48)^3]_C$$

$$I_y = 64.3\text{x}10^3 \text{ mm}^4$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{64.3\times 10^3}{[(24\times 6)+(8\times 48)+(48\times 6)]}} = 8.87\, mm$$

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.





$$I_x = (\bar{I}_x + \cancelto{0}{Ad_y^{\,2}})_{A5\times6} - (\bar{I}_x + Ad_y^{\,2})_{B4\times2} - (\bar{I}_x + Ad_y^{\,2})_{C4\times1}$$

$$= [\frac{1}{12}(5)(6)^3 + 0]_A - [\frac{1}{12}(4)(2)^3 + (2\times4)(2)^2]_B$$

$$-[\frac{1}{12}(4)(1)^3 + (4\times1)(1.5)^2]_C$$

$$I_x = 46 \text{ m}^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{46}{[(5\times6)-(4\times2)-(4\times1)]}} = 1.599\, m$$

$$I_y = (\bar{I}_y + \cancelto{0}{Ad_x^{\,2}})_A - (\bar{I}_y + \cancelto{0}{Ad_x^{\,2}})_B - (\bar{I}_y + \cancelto{0}{Ad_x^{\,2}})_C$$

$$= [\frac{1}{12}(6)(5)^3]_A - [\frac{1}{12}(2)(4)^3]_B - [\frac{1}{12}(1)(4)^3]_C$$

$$I_y = 46.5 \text{ m}^4$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{46.5}{[(5\times6)-(4\times2)-(4\times1)]}} = 1.607\, m$$