

Engineering Mechanics

First Class

Syllabus

1. Resultant of force systems, resultant of concurrent force system.
2. Component of force, moment and couple.
3. Resultant of non-concurrent force system.
4. Equilibrium of force systems, equations of equilibrium.
5. Free body diagram.
6. Centroids and center of areas.
7. Equations and applications.
8. Moments of inertia.
9. Transfer formula of moment of inertia.
10. Types of supports and loads.
11. Analysis of trusses, pinned trusses.
12. Method of joints.
13. Method of section.
14. Simple stresses, axial stress.
15. Shear stress.
16. Bearing stress.

Engineering Mechanics

First Class

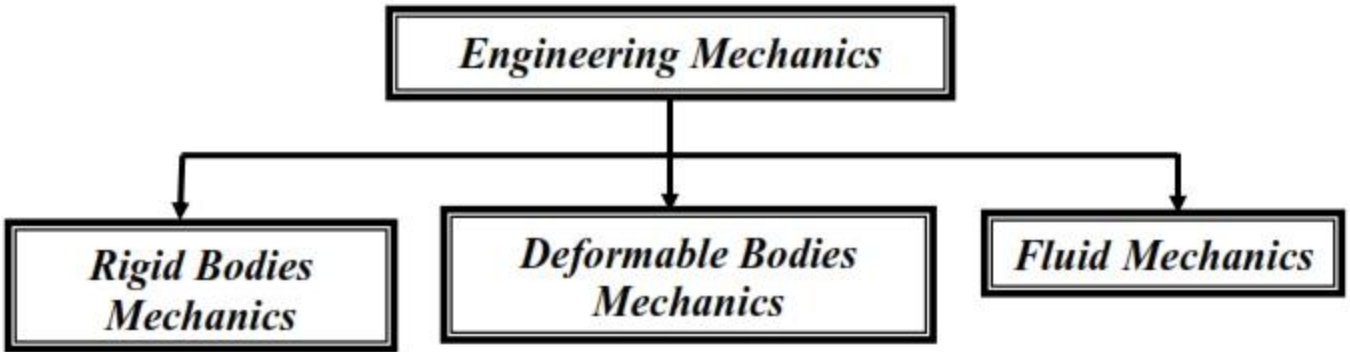
Static: Force system, Units system, Forces + Components, Resultant, Moment and Couples, Equilibrium, Centroid, Moment of Inertia, Friction.

Dynamics: Rectilinear motion, Curvilinear motion, Projectile, Circular motion, Acceleration Components (Rectangular Comp., Normal Tangential Comp.), Kinetic - 2nd Law of Newton.

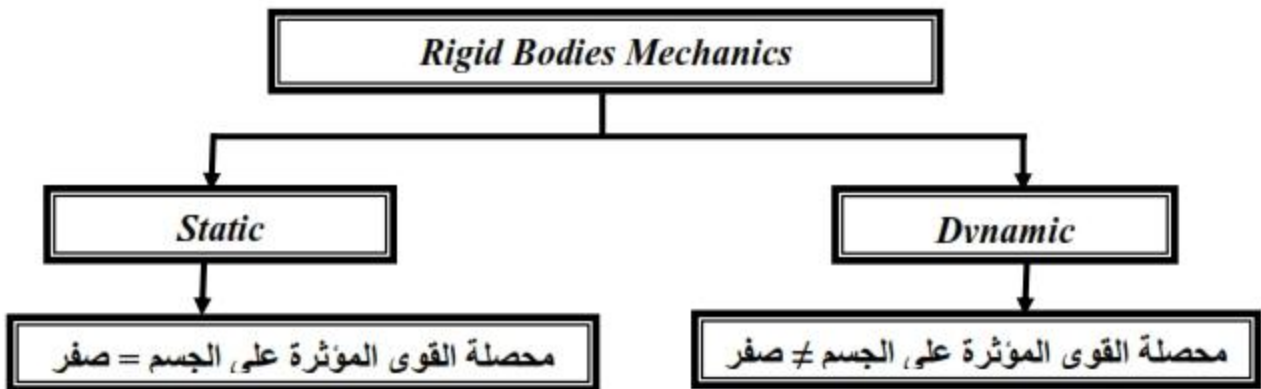
1. Introduction:

Mechanics: can be defined as that branch of the physical sciences concerned with the state of bodies that are subjected to the action of forces (in the state of motion or rest)

الميكانيك يعرف بأنه جزء من العلوم الفيزيائية التي تركز على حالة الاجسام تحت تأثير القوى الخارجية المسلطة على هذه الاجسام.



- When the changes in shape of body are important, the problem becomes **Deformable Bodies Mechanics**.
- Our study treats only with **Rigid Bodies Mechanics**, so that the body is stay in the same shape after applying the forces (No deformations are considered in the body). (در استتنا سوف تكون على الاجسام الغير قابلة للتشوه (الاجسام الجاسنة)).



- **Static** deals with the equilibrium of bodies. That are either at rest or move with a constant velocity. الاجسام سوف تكون في حالة توازن اما ساكنة او متحركة بسرعة ثابتة
- **Dynamic** is concerned with the accelerated motion of bodies under effects of external forces. الاجسام سوف تكون في حالة حركة متغيرة.

Vector & Scalar quantities :

Vector quantities : are the quantities which have magnitude and direction .such as: Force , weight , distance , speed , displacement , acceleration ,velocity .

Scalar quantities : are the quantities which have only magnitude , such as : Time , size , sound , density , light , volume .

Force :

A "**force**" is an action that changes, or tends to change, the state of motion of the body upon which it acts. It is a vector quantity that can be represented either mathematically or graphically

A complete description of a force MUST include its:

1. MAGNITUDE
2. DIRECTION and SENSE
3. POINT OF ACTION

هي الفعل الذي يغير او يحاول ان يغير من حالة الجسم الحركية او الشكلية، والقوة لها مقدار (Magnitude) واتجاه (Direction) ووجهة (Sense) ونقطة تأثير (Action point)

▪ **Classification of Forces**

✓ Contact

- 1 – Contacting or surface forces (mechanical)
- 2 – Non-Contacting or body forces (gravitational, weight)

✓ Area

- 1 – Distributed Force, uniform and non-uniform
- 2 – Concentrated Force

▪ **Classification of Forces**

✓ Force System

- 1 – Concurrent : all forces pass through a point
- 2 – Coplanar : in the same plane
- 3 – Parallel : parallel line of action
- 4 – Collinear : common line of action

✓ Three Types

- 1 – Free (direction, magnitude and sense)
- 2 – Sliding
- 3 – Fixed

Newton's Laws

Sir Isaac Newton was the first to state correctly the basic laws governing the motion of a particle and to demonstrate their validity.* Slightly reworded with modern terminology, these laws are:

Law I. A particle remains at rest or continues to move with *uniform velocity* (in a straight line with a constant speed) if there is no unbalanced force acting on it.

Law II. The acceleration of a particle is proportional to the vector sum of forces acting on it, and is in the direction of this vector sum.

Law III. The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and *collinear* (they lie on the same line).

The correctness of these laws has been verified by innumerable accurate physical measurements. Newton's second law forms the basis for most of the analysis in dynamics. As applied to a particle of mass m , it may be stated as

$$\mathbf{F} = m\mathbf{a}$$

(1/1)

where \mathbf{F} is the vector sum of forces acting on the particle and \mathbf{a} is the resulting acceleration. This equation is a *vector* equation because the direction of \mathbf{F} must agree with the direction of \mathbf{a} , and the magnitudes of \mathbf{F} and $m\mathbf{a}$ must be equal.

Newton's first law contains the principle of the equilibrium of forces, which is the main topic of concern in statics. This law is actually a consequence of the second law, since there is no acceleration when the force is zero, and the particle either is at rest or is moving with a uniform velocity. The first law adds nothing new to the description of motion but is included here because it was part of Newton's classical statements.

The third law is basic to our understanding of force. It states that forces always occur in pairs of equal and opposite forces. Thus, the downward force exerted on the desk by the pencil is accompanied by an upward force of equal magnitude exerted on the pencil by the desk. This principle holds for all forces, variable or constant, regardless of their source, and holds at every instant of time during which the forces are applied. Lack of careful attention to this basic law is the cause of frequent error by the beginner.

In the analysis of bodies under the action of forces, it is absolutely necessary to be clear about which force of each action-reaction pair is being considered. It is necessary first of all to *isolate* the body under consideration and then to consider only the one force of the pair which acts *on* the body in question.

Unites and Their Relations:

QUANTITY	DIMENSIONAL SYMBOL	SI UNITS		U.S. CUSTOMARY UNITS		
		UNIT	SYMBOL	UNIT	SYMBOL	
Mass	M	Base units	kilogram	kg	Base units	slug
Length	L		meter	m		foot
Time	T		second	s		second
Force	F		newton	N		pound

1 m	100 cm
1 in	2.54 cm
1 m	1000 mm
1 ft	12 in
1 km	1000 m
1 mile	1609.1 m
1 yard	3 ft
1 kg	2.204 lb (pound)
1 kg	9.81 N
1 ton	1000 kg

SI Unites

The International System of Units, abbreviated SI (from the French, *Système International d'Unités*), is accepted in the United States and throughout the world, and is a modern version of the metric system. By international agreement, SI units will in time replace other systems. As shown in the table, in SI, the units kilogram (kg) for mass, meter (m) for length, and second (s) for time are selected as the base units, and the newton (N) for force is derived from the preceding three by Eq. 1/1. Thus, force (N) = mass (kg) \times acceleration (m/s^2) or

$$N = kg \cdot m/s^2$$

Thus, 1 newton is the force required to give a mass of 1 kg an acceleration of $1 m/s^2$.

Consider a body of mass m which is allowed to fall freely near the surface of the earth. With only the force of gravitation acting on the body, it falls with an acceleration g toward the center of the earth. This gravitational force is the *weight* W of the body, and is found from Eq. 1/1:

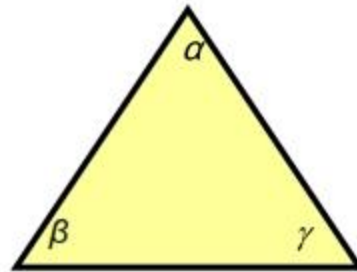
$$W(N) = m(kg) \times g(m/s^2)$$

Quantity Measurement Units:

Area	-----	m^2, cm^2
Length	-----	m, cm
Volume	-----	m^3, cm^3
Mass	-----	kg
Force	-----	N, KN
Moment	-----	$N.m, kN.m$
Time	-----	$sec, min, hr.$
Angle	-----	$degree, radian$

Symbols:

α	ALPPHA
β	BETA
γ	GAMMA
ϕ	PHI
π	PI
μ	MU



Trigonometric Relations for Right Angle's Triangles

$$\sin \beta = \frac{a}{c}$$

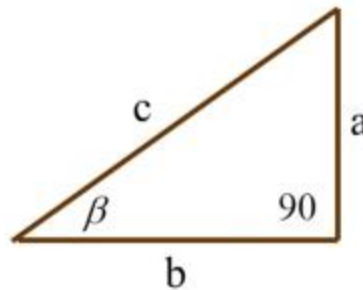
$$\cos \beta = \frac{b}{c}$$

$$\tan \beta = \frac{a}{b}$$

$$\sec \beta = \frac{c}{b} = 1 / \cos \beta$$

$$\cot \beta = \frac{b}{a} = 1 / \tan \beta$$

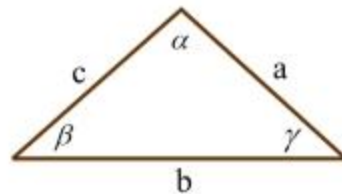
$$c \sec \beta = \frac{c}{a} = 1 / \sin \beta$$



Oblique Triangle

a-Sine Low

$$\frac{a}{\sin \beta} = \frac{b}{\sin \alpha} = \frac{c}{\sin \gamma}$$



b-Cosine Low

$$a^2 = b^2 + c^2 - 2bc \cos \beta$$

$$b^2 = a^2 + c^2 - 2ac \cos \alpha$$

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

General Trigonometric Formulas

$$\sin \alpha = 2 \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha$$

$$\cos \alpha = 2 \cos^2 \frac{1}{2} \alpha - 1 = 1 - \sin^2 \frac{1}{2} \alpha$$

$$\cos \alpha = \cos^2 \frac{1}{2} \alpha - \sin^2 \frac{1}{2} \alpha$$

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\sin 2\alpha}{1 + \cos 2\alpha} = \sqrt{\sec^2 \alpha - 1} = \sqrt{\cos^2 \alpha + \sin^2 \alpha}$$

2. Composition and Resolution of Force :

Let the force (F) shown in fig.(1) with the direction (θ) We can resolve this force into two components :

- 1- Horizontal component (F_x) which lies on x- axis
- 2- Vertical component (F_y) which lies on y- axis as shown in fig.(2)

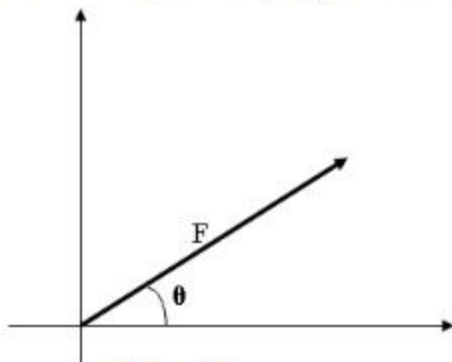


Fig. (1)

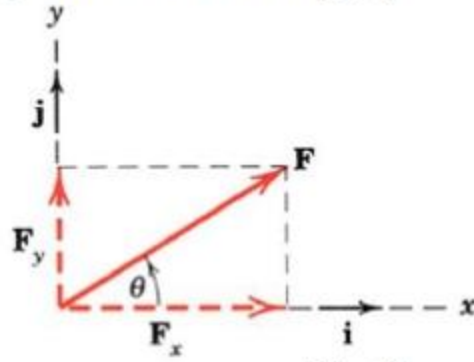


Fig.(2)

Thus from fig.(2) :

The horizontal component may be determined as :

$$F_x = F \cdot \cos \theta$$

The vertical component may be determined as

$$F_y = F \cdot \sin \theta$$

$$F_x = F \cos \theta \quad F = \sqrt{F_x^2 + F_y^2}$$

$$F_y = F \sin \theta \quad \theta = \tan^{-1} \frac{F_y}{F_x}$$

EX (1) :

Find the two components of the force (100 N) if : $\theta = 30^\circ$, 120° , 270°
fig. (2)

Solution :

$\theta = 30^\circ$:

$$F_x = F \cdot \cos \theta$$

$$= 100 \cdot \cos 30$$

$$= 100 \cdot \frac{\sqrt{3}}{2} = 50\sqrt{3} \text{ N}$$

$$F_y = F \cdot \sin \theta$$

$$= 100 \cdot \sin 30$$

$$= 100 \cdot 0.5 = 50 \text{ N}$$

$\theta = 120^\circ$:

$$F_x = F \cdot \cos \theta$$

$$= 100 \cdot \cos 120$$

$$= 100 \cdot (-0.5) = -50 \text{ N}$$

$$F_y = F \cdot \sin \theta$$

$$= 100 \cdot \sin 120$$

$$= 100 \cdot \frac{\sqrt{3}}{2} = 50\sqrt{3} \text{ N}$$

$\theta = 270^\circ$:

$$F_x = F \cdot \cos \theta$$

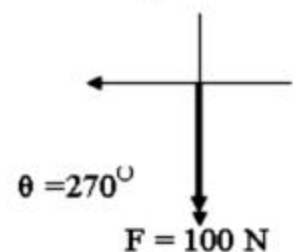
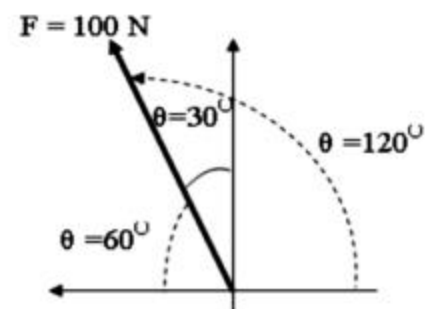
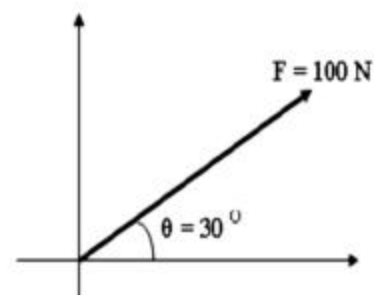
$$= 100 \cdot \cos 270$$

$$= 100 \cdot (0) = 0$$

$$F_y = F \cdot \sin \theta$$

$$= 100 \cdot \sin 270$$

$$= 100 \cdot (-1) = -100 \text{ N}$$



EX (2):

The forces F_1 , F_2 , and F_3 , all of which act on point A of the bracket, are specified in three different ways. Determine the x and y scalar components of each of the three forces.

Solution. The scalar components of F_1 , from Fig. a, are

$$F_{1x} = 600 \cos 35^\circ = 491 \text{ N} \quad \text{Ans.}$$

$$F_{1y} = 600 \sin 35^\circ = 344 \text{ N} \quad \text{Ans.}$$

The scalar components of F_2 , from Fig. b, are

$$F_{2x} = -500\left(\frac{4}{5}\right) = -400 \text{ N} \quad \text{Ans.}$$

$$F_{2y} = 500\left(\frac{3}{5}\right) = 300 \text{ N} \quad \text{Ans.}$$

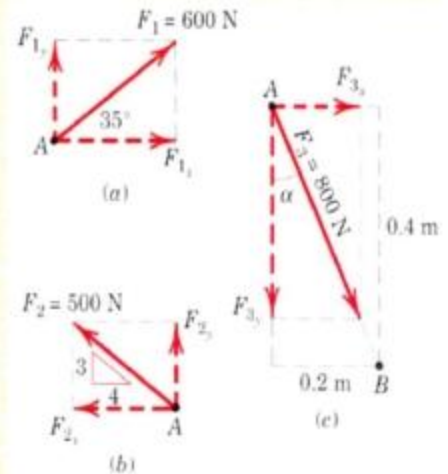
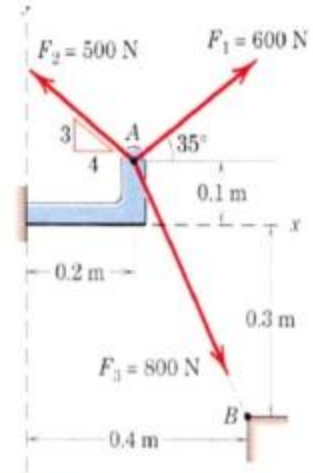
Note that the angle which orients F_2 to the x-axis is never calculated. The cosine and sine of the angle are available by inspection of the 3-4-5 triangle. Also note that the x scalar component of F_2 is negative by inspection.

The scalar components of F_3 can be obtained by first computing the angle α of Fig. c.

$$\alpha = \tan^{-1} \left[\frac{0.2}{0.4} \right] = 26.6^\circ$$

① Then $F_{3x} = F_3 \sin \alpha = 800 \sin 26.6^\circ = 358 \text{ N} \quad \text{Ans.}$

$$F_{3y} = -F_3 \cos \alpha = -800 \cos 26.6^\circ = -716 \text{ N} \quad \text{Ans.}$$



EX(3)

The direction of the force (P) is (30°), Find the horizontal component if the vertical component is (30 N) ?

Solution :

From the diagram shown :

$$F_y = 30 \text{ N}$$

$$F_y = F \cdot \sin \theta$$

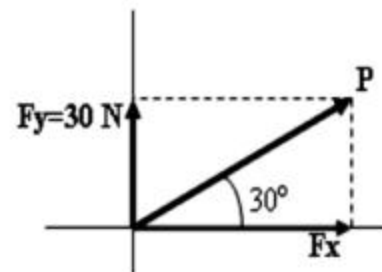
$$30 = P \cdot \sin 30$$

$$30 = P \cdot 0.5$$

$$P = 30 / 0.5 = 60 \text{ N}$$

$$F_x = F \cdot \cos \theta$$

$$= 60 \cdot \cos 30 = 60 \cdot \frac{\sqrt{3}}{2} = 30\sqrt{3} \text{ N}$$



Composition of Force :

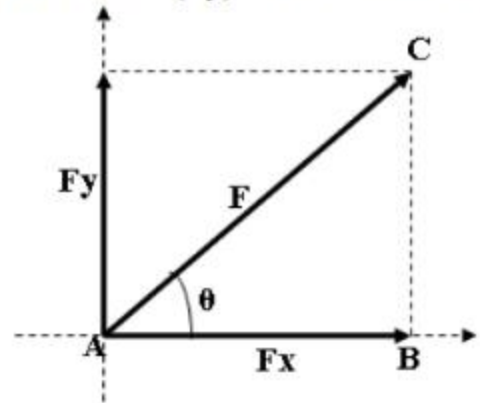
Let we have (F_x) is the horizontal component and (F_y) is the vertical component for the force (F) shown in fig.

From the shape ABC we get :

$$AC^2 = AB^2 + BC^2$$

$$F^2 = F_x^2 + F_y^2$$

$$F = \sqrt{(F_x)^2 + (F_y)^2}$$



Determination of the direction of force :

The direction of a force can be determined by :

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)$$

Ex (4) :

Determine the magnitude and direction of a force (P) , if the horizontal and vertical components are (20 N) , (40 N) respectively ?

Solution :

We have : $F_x = 20 \text{ N}$, $F_y = 40 \text{ N}$, $F = \sqrt{(F_x)^2 + (F_y)^2}$


$$F = \sqrt{(20)^2 + (40)^2} = \sqrt{400 + 1600} = \sqrt{20000} = 44.72 \text{ N}$$

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{40}{20}\right) = 63.43^\circ$$

Resultant of forces system

The resultant is a representative force which has the same effect on the body as the group of forces it replaces.

A simplest force which can replace the original forces system without changing its external effect on a rigid body.

The symbol of resultant force is: 

The unit of resultant force is : Newton (N)

The resultant is applied for different types of forces system as:

1- Coplanar forces system :

a- concurrent coplanar forces system (متلاقية في مستو واحد)

b- non-concurrent coplanar forces system (غير متلاقية في مستو واحد)

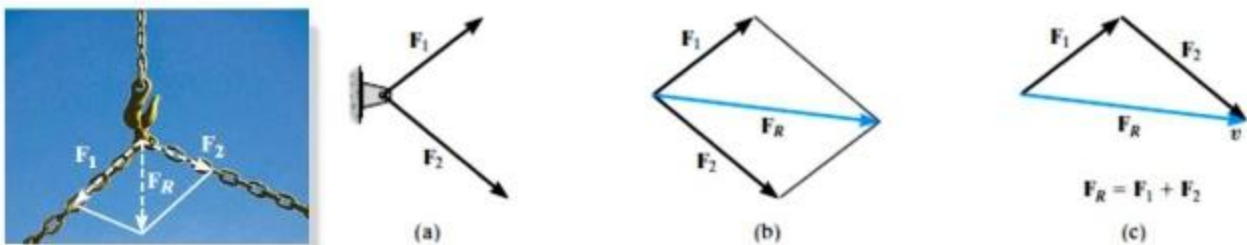
2- Non coplanar forces system :

a- concurrent non-coplanar forces system (متلاقية ليست في مستو واحد)

b- non-concurrent non-coplanar forces system (غير متلاقية ليست في مستو واحد)

Resultant of concurrent coplanar forces system

Finding a Resultant Force. The two component forces F_1 and F_2 acting on the pin in Fig. *a* can be added together to form the resultant force $F_R = F_1 + F_2$, as shown in Fig. *b*. From this construction, or using the triangle rule, Fig. *c*, we can apply the law of cosines or the law of sines to the triangle in order to obtain the magnitude of the resultant force and its direction.



We will find out the resultant force for many forces acting on a rigid body by using the following equations

$$R_x = F_1 \cdot \cos \theta_1 + F_2 \cdot \cos \theta_2 + F_3 \cdot \cos \theta_3 + \dots + F_n \cdot \cos \theta_n$$

$$R_y = F_1 \cdot \sin \theta_1 + F_2 \cdot \sin \theta_2 + F_3 \cdot \sin \theta_3 + \dots + F_n \cdot \sin \theta_n$$

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

The direction of resultant force may be determined as :

$$\theta = \tan^{-1} \left(\frac{R_y}{R_x} \right)$$

Ex (5)

Find the resultant force for the concurrent coplanar forces system, shown in figure.

Solution:

$$R_x = F_1 \cdot \cos \theta_1 \mp F_2 \cdot \cos \theta_2 \mp F_3 \cdot \cos \theta_3$$

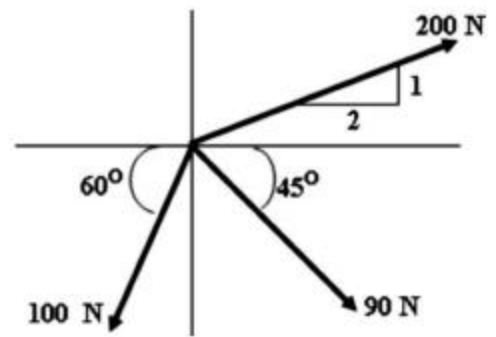
$$= 200 \cdot \frac{2}{\sqrt{5}} - 100 \cos 60 + 90 \cos 45 = +192.4 \text{ N}$$

$$R_y = F_1 \cdot \sin \theta_1 \mp F_2 \cdot \sin \theta_2 \mp F_3 \cdot \sin \theta_3$$

$$= 200 \cdot \frac{1}{\sqrt{5}} - 100 \sin 60 - 90 \sin 45 = -60.8 \text{ N}$$

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

$$= \sqrt{(192.4)^2 + (60.8)^2} = 202 \text{ N}$$

**Ex (6):**

Determine the resultant force for the forces system shown in fig.

Solution:

$$R_x = F_1 \cdot \cos \theta_1 \mp F_2 \cdot \cos \theta_2 \mp F_3 \cdot \cos \theta_3$$

$$= 100 \cos 90 + 250 \cos(0) - 200 \cos 45$$

$$= 192.5 \text{ N}$$

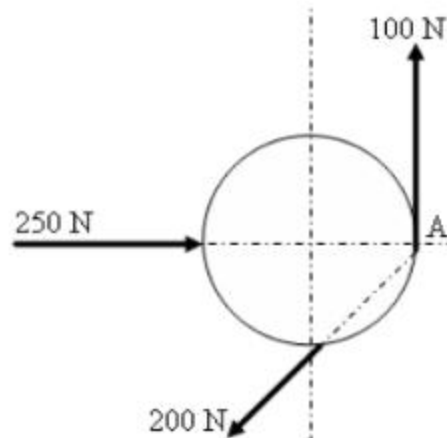
$$R_y = F_1 \cdot \sin \theta_1 \mp F_2 \cdot \sin \theta_2 \mp F_3 \cdot \sin \theta_3$$

$$= 100 \sin 90 + 250 \sin(0) - 200 \sin 45$$

$$= -60.78 \text{ N}$$

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

$$= \sqrt{(192.5)^2 + (-60.78)^2} = 201.8 \text{ N}$$

**Ex (7):**

The 1000 N force is a resultant of two forces, one of which is 600 N, Determine the other force?

Solution:

$$R_x = F_1 \cdot \cos \theta_1 \mp F_2 \cdot \cos \theta_2$$

$$R \cdot \cos \theta_R = F_1 \cdot \cos \theta_1 \mp F_2 \cdot \cos \theta_2$$

$$-1000 \cdot \cos 60 = 600 \cdot \frac{3}{5} + F_2 \cos \theta_2$$

$$-1000 \cdot 0.5 = 360 + F_2 \cos \theta_2$$

$$F_2 \cos \theta_2 = -860 \text{ N} \quad \text{_____ (1)} \Rightarrow F_x$$

$$R_y = F_1 \cdot \sin \theta_1 \mp F_2 \cdot \sin \theta_2$$

$$R \cdot \sin \theta_R = F_1 \cdot \sin \theta_1 \mp F_2 \cdot \sin \theta_2$$

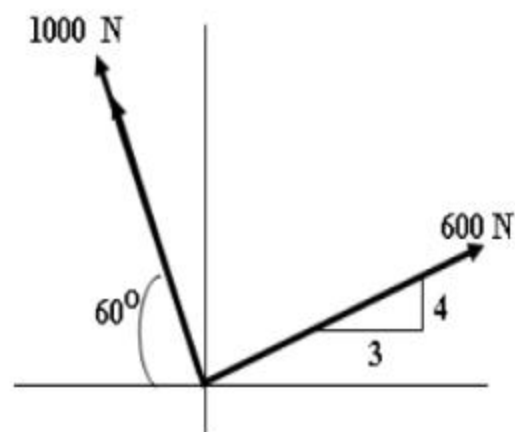
$$1000 \cdot \sin 60 = 600 \cdot \frac{4}{5} + F_2 \sin \theta_2$$

$$F_2 \sin \theta_2 = 386.02 \text{ N} \quad \text{_____ (2)} \Rightarrow F_y$$

$$F = \sqrt{(F_x)^2 + (F_y)^2}$$

$$= \sqrt{(860)^2 + (386.02)^2}$$

$$= 942.62 \text{ N}$$



Parallelogram Law

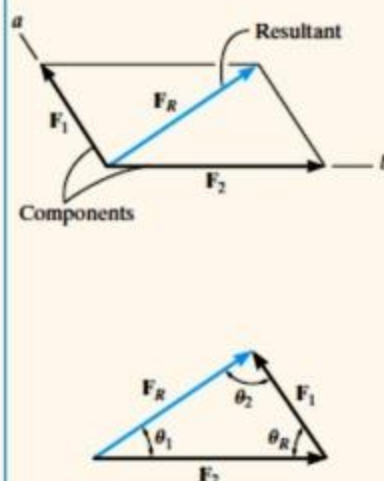
Two forces add according to the parallelogram law. The *components* form the sides of the parallelogram and the *resultant* is the diagonal.

To find the components of a force along any two axes, extend lines from the head of the force, parallel to the axes, to form the components.

To obtain the components or the resultant, show how the forces add by tip-to-tail using the triangle rule, and then use the law of cosines and the law of sines to calculate their values.

$$F_R = \sqrt{F_1^2 + F_2^2 - 2 F_1 F_2 \cos \theta_R}$$

$$\frac{F_1}{\sin \theta_1} = \frac{F_2}{\sin \theta_2} = \frac{F_R}{\sin \theta_R}$$

**Ex(8):**

Determine the angle (θ) so that the resultant (R) of the two forces system shown in fig. is 100 N

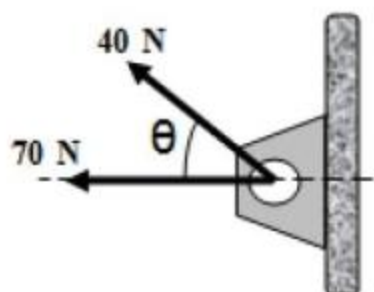
Solution

$$R = \sqrt{(F_1)^2 + (F_2)^2 + 2F_1F_2 \cos \theta}$$

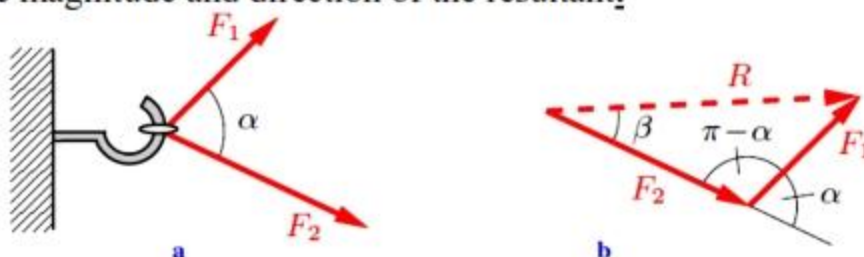
$$100 = \sqrt{(70)^2 + (40)^2 + 2 * 70 * 40 * \cos \theta}$$

$$\cos \theta = 0.625$$

$$\theta = 51.3^\circ$$



Ex (9): A hook carries two forces F_1 and F_2 , which define the angle α (Fig.a). Determine the magnitude and direction of the resultant.



Solution: Since the problem will be solved by trigonometry (and since the magnitudes of the forces are not given numerically), a sketch of the force triangle is drawn, but not to scale (Fig.b). We assume that the magnitudes of the forces F_1 and F_2 and the angle α are known quantities in this force plan. Then the magnitude of the resultant follows from the law of cosines:

$$R^2 = F_1^2 + F_2^2 - 2 F_1 F_2 \cos (\pi - \alpha)$$

Or
$$R = \sqrt{F_1^2 + F_2^2 + 2 F_1 F_2 \cos \alpha} .$$

The angle β gives the direction of the resultant R with respect to the force F_2 (Fig.b).

The law of sines yields

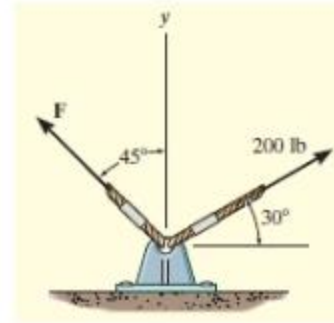
$$\frac{\sin \beta}{\sin (\pi - \alpha)} = \frac{F_1}{R}$$

Introducing the result for R and using the trigonometrical relation $\sin(\pi - \alpha) = \sin \alpha$ we obtain

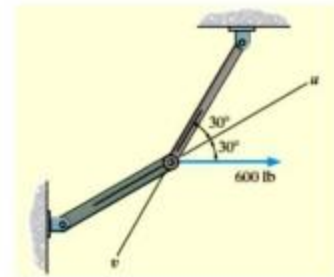
$$\sin \beta = \frac{F_1 \sin \alpha}{\sqrt{F_1^2 + F_2^2 + 2 F_1 F_2 \cos \alpha}}$$

Problems:

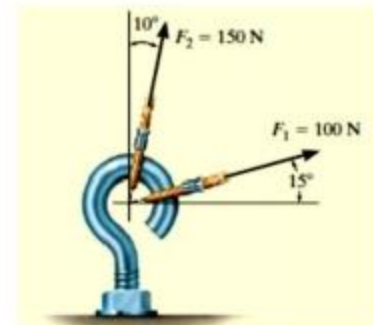
Q1.1: Determine the magnitude of the component force F in Figure and the magnitude of the resultant force F_R if F_R is directed along the positive y axis.



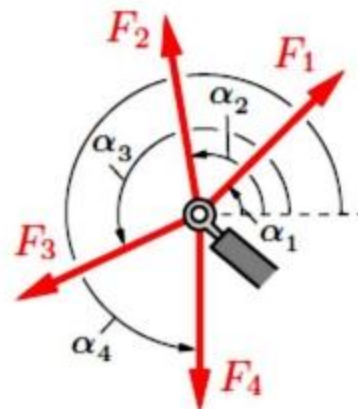
Q1.2: Resolve the horizontal 600-lb force in Figure into components acting along the u and v axes and determine the magnitudes of these components



Q1.3 The screw eye in Figure below subjected to two forces, F_1 and F_2 . Determine the magnitude and direction of the resultant force.



Q1.4 An eyebolt is subjected to four forces ($F_1=12\text{kN}$, $F_2=8\text{kN}$, $F_3=18\text{kN}$, $F_4=4\text{kN}$) that act under given angles ($\alpha_1 = 45^\circ$, $\alpha_2=100^\circ$, $\alpha_3= 205^\circ$, $\alpha_4=270^\circ$) with respect to the horizontal (Figure). Determine the magnitude and direction of the resultant.



3. Moment of Force

The moment of a force: is the ability of the force to produce turning or twisting about an axis or point or line.

Mathematically:

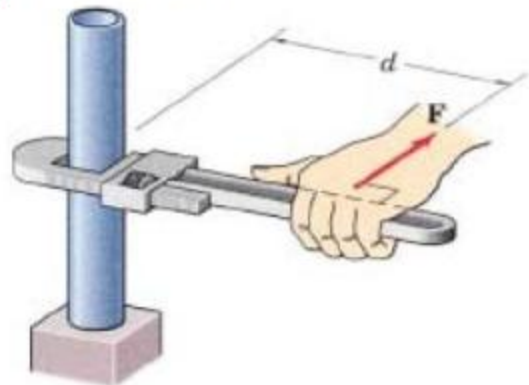
The moment of a force = the applied force \times perpendicular distance

$$M = F * d$$

M = the moment of a force (N.m)

F = applied force (N)

d = perpendicular distance between the point of action of the force and moment center.



Ex (1) :

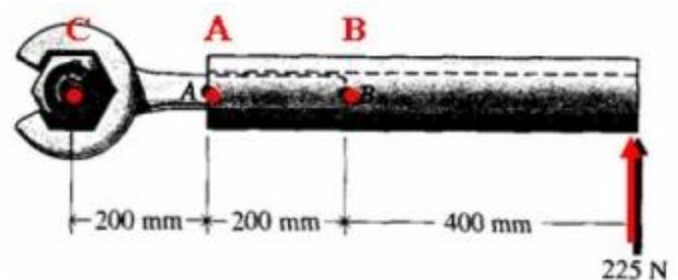
Determine the moment of the force 225 N about the Points A , B , and C .

Solution:

$$MA = |F| dA = 225 * 0.6 = 135 \text{ Nm}$$

$$MB = |F| dB = 225 * 0.4 = 90 \text{ Nm}$$

$$MC = |F| dC = 225 * 0.8 = 180 \text{ Nm}$$



Ex (2) :

Determine the moment of the force 500 N about the point A and B .

Solution:

$$\cos(60) = 200/L$$

$$\cos(60) = d_{ac}/(L-160)$$

$$L = 200/\cos(60)$$

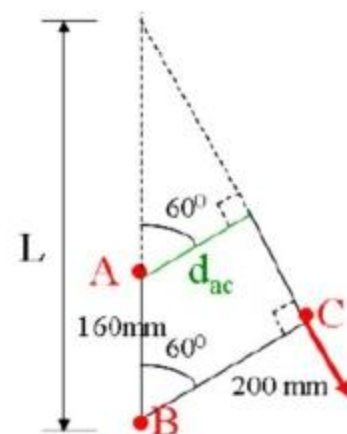
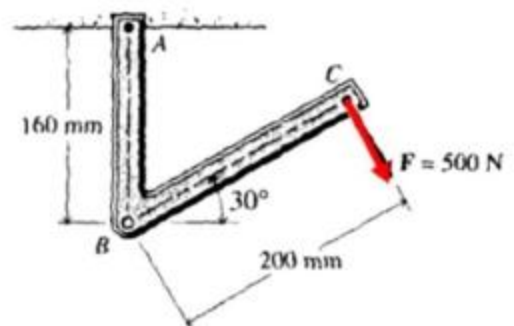
$$L = 160 + d_{ac}/\cos(60)$$

$$d_{ac} = 200 - 160 \cos(60) = 120 \text{ mm}$$

$$d_{ac} = 120 \text{ mm}$$

$$MA = |F| d_{AC} = 500 * 0.12 = 60 \text{ Nm}$$

$$MB = |F| dB = 500 * 0.2 = 100 \text{ Nm}$$



Ex (3) :

Find the moment of the force **200 N** About the point **(A)** shown in fig.

Solution

$$F_x = F \cdot \cos \theta = 200 \cos 45$$

$$= 200 * 0.707 = 141.42 \text{ N}$$

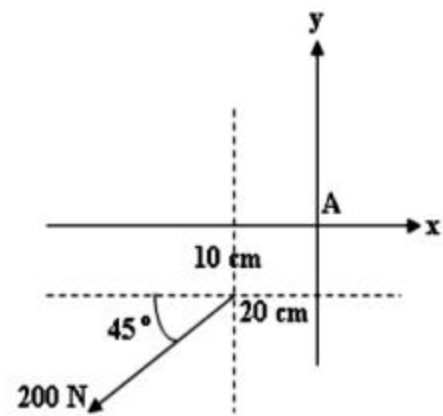
$$F_y = F \cdot \sin \theta = 200 \sin 45$$

$$= 200 * 0.707 = 141.42 \text{ N}$$

$$M_1 = F_x * d = 141.42 * 10 = 1414.2 \text{ N} \cdot \text{cm}$$

$$M_2 = F_y * d = 141.42 * 20 = 2828.4 \text{ N} \cdot \text{cm}$$

$$M (A) = M_1 - M_2 = - 1414.2 \text{ N} \cdot \text{cm}$$

**Ex (4) :**

Determine the moment of the force **(70 N)** shown in fig. about the Point **(A)** .

Solution

$$F_x = F \cdot \cos \theta = 70 \cos 30$$

$$= 70 * 0.866 = 60.62 \text{ N}$$

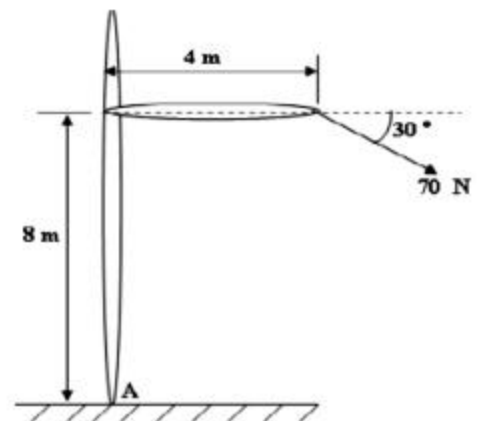
$$F_y = F \cdot \sin \theta = 70 \sin 30$$

$$= 70 * 0.5 = 35 \text{ N}$$

$$M_1 = F_x * d = 60.62 * 8 = 484.97 \text{ N} \cdot \text{m}$$

$$M_2 = F_y * d = 35 * 4 = 140 \text{ N} \cdot \text{m}$$

$$M (A) = M_1 + M_2 = 484.97 \text{ N} + 140 = 624.97 \text{ N} \cdot \text{m}$$

**Ex (5) :**

Find the distance **(Xn)** , if the moment of the force **(F)** about the point **(A)** is equal to **zero** .

Solution

$$F_x = F \cdot \cos \theta = 20 \cos 30$$

$$= 20 * 0.866 = 17.32 \text{ N}$$

$$F_y = F \cdot \sin \theta = 20 \sin 30$$

$$= 20 * 0.5 = 10 \text{ N}$$

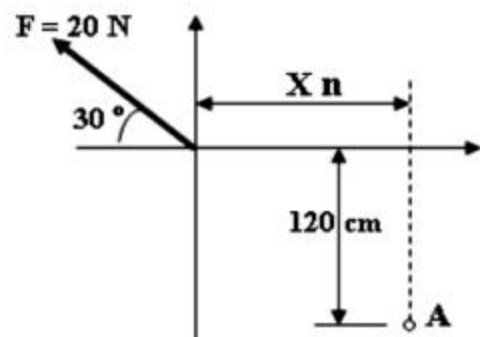
$$M_1 = F_x * d = 17.32 * 120 = - 2078.46 \text{ N} \cdot \text{cm}$$

$$M_2 = F_y * d = 10 * X_n = 10 X_n \text{ N} \cdot \text{cm}$$

$$M (A) = - M_1 + M_2$$

$$0 = - 2078.46 + 10 X_n$$

$$X_n = 2078.46 / 10 = 207.84 \text{ cm}$$



Ex (6) :

The moment of the force (F) about (A) is ($100\sqrt{3}$) $N \cdot m$), the moment of this force about (B) is ($-50\sqrt{3}$ $N \cdot m$), the distance between (A) and (B) is (30 m), Find the **magnitude** and the **location** of the force (F) if its component along the line (AB) is (5 N).

Solution

Given $F_x = 5$ N

$$M(A) = 100\sqrt{3}, M(B) = -50\sqrt{3}$$

$$M(A) = F \cdot \sin \theta \cdot x$$

$$F \cdot \sin \theta = 100\sqrt{3} / x$$

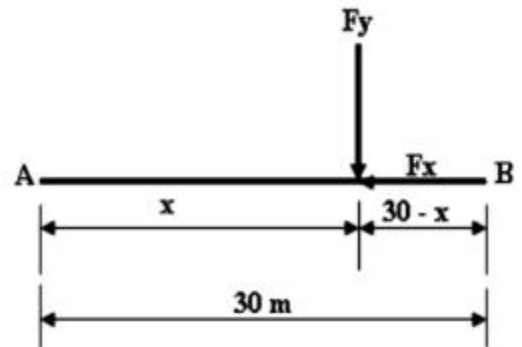
$$M(B) = F \cdot \sin \theta \cdot (30 - x)$$

$$F \cdot \sin \theta = -50\sqrt{3} / (30 - x)$$

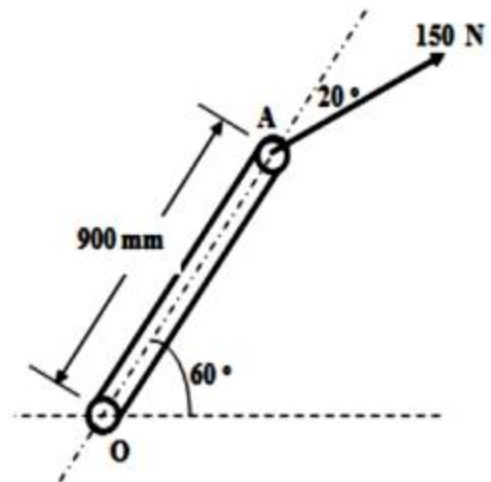
$$\frac{100\sqrt{3}}{x} = \frac{50\sqrt{3}}{30 - x} \implies X = 20 \text{ m}$$

$$F_y = F \cdot \sin \theta = 100\sqrt{3} / x = 100\sqrt{3} / 20 = 5\sqrt{3} \text{ N}$$

$$F = \sqrt{(F_x)^2 + (F_y)^2} = \sqrt{(5)^2 + (5\sqrt{3})^2} = 10 \text{ N}$$

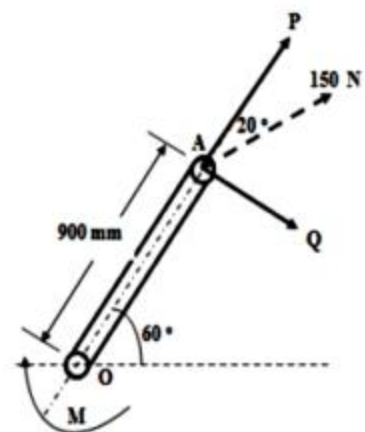
**Ex (7) :**

A (150 N) force acts on the end of the (900 mm) lever as shown in fig. Determine the moment of the force about (O).

**Solution**

$$Q = 150 \sin 20 = 51.3 \text{ N}$$

$$M_o = -Q(0.9) = -51.3 \cdot 0.9 = -46.2 \text{ N.m} = 64.2 \text{ N.m}$$



Ex(8):

Example Problem 8.2

Calculate the magnitude of the moment about the base point O of the 600-N force in five different ways.

Solution. (I) The moment arm to the 600-N force is

$$d = 4 \cos 40^\circ + 2 \sin 40^\circ = 4.35 \text{ m}$$

① By $M = Fd$ the moment is clockwise and has the magnitude

$$M_O = 600(4.35) = 2610 \text{ N}\cdot\text{m}$$

Ans.

(II) Replace the force by its rectangular components at A

$$F_1 = 600 \cos 40^\circ = 460 \text{ N}, \quad F_2 = 600 \sin 40^\circ = 386 \text{ N}$$

By Varignon's theorem, the moment becomes

② $M_O = 460(4) + 386(2) = 2610 \text{ N}\cdot\text{m}$

Ans.

(III) By the principle of transmissibility, move the 600-N force along its line of action to point B , which eliminates the moment of the component F_2 . The moment arm of F_1 becomes

$$d_1 = 4 + 2 \tan 40^\circ = 5.68 \text{ m}$$

and the moment is

$$M_O = 460(5.68) = 2610 \text{ N}\cdot\text{m}$$

Ans.

③ (IV) Moving the force to point C eliminates the moment of the component F_1 . The moment arm of F_2 becomes

$$d_2 = 2 + 4 \cot 40^\circ = 6.77 \text{ m}$$

and the moment is

$$M_O = 386(6.77) = 2610 \text{ N}\cdot\text{m}$$

Ans.

(V) By the vector expression for a moment, and by using the coordinate system indicated on the figure together with the procedures for evaluating cross products, we have

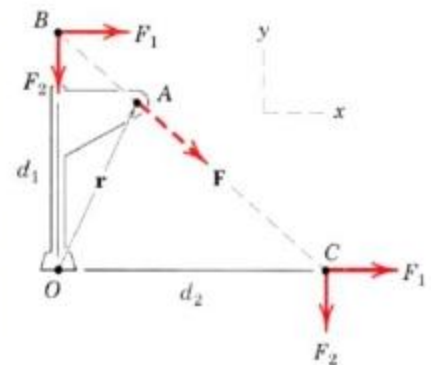
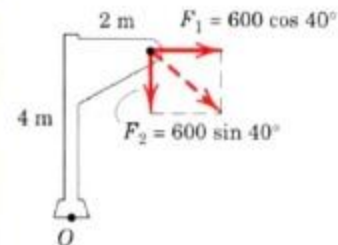
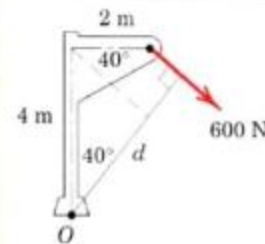
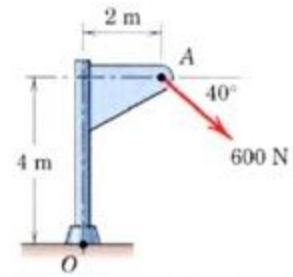
④
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = (2\mathbf{i} + 4\mathbf{j}) \times 600(\mathbf{i} \cos 40^\circ - \mathbf{j} \sin 40^\circ)$$

$$= -2610\mathbf{k} \text{ N}\cdot\text{m}$$

The minus sign indicates that the vector is in the negative z -direction. The magnitude of the vector expression is

$$M_O = 2610 \text{ N}\cdot\text{m}$$

Ans.



Helpful Hints

- ① The required geometry here and in similar problems should not cause difficulty if the sketch is carefully drawn.
- ② This procedure is frequently the shortest approach.
- ③ The fact that points B and C are not on the body proper should not cause concern, as the mathematical calculation of the moment of a force does not require that the force be on the body.
- ④ Alternative choices for the position vector \mathbf{r} are $\mathbf{r} = d_1\mathbf{j} = 5.68\mathbf{j} \text{ m}$ and $\mathbf{r} = d_2\mathbf{i} = 6.77\mathbf{i} \text{ m}$.

- **Couples**

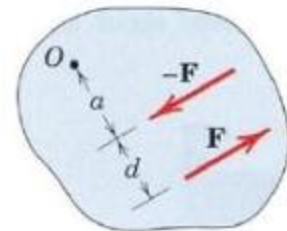
A special case of moments is a couple. A **couple** consists of two parallel forces that are equal in magnitude, opposite in sense and do not share a line of action. It does not produce any translation, only rotation. The resultant force of a couple is zero. BUT, the resultant of a couple is not zero; it is a pure moment.

Consider the action of two equal and opposite forces \mathbf{F} and $-\mathbf{F}$ a distance d apart, as shown in the Figure. These two forces cannot be combined into a single force because their sum in every direction is zero. Their only effect is to produce a tendency of rotation. The combined moment of the two forces about an axis normal to their plane and passing through any point such as O in their plane is the couple \mathbf{M} . This couple has a magnitude

$$M = F(a + d) - Fa$$

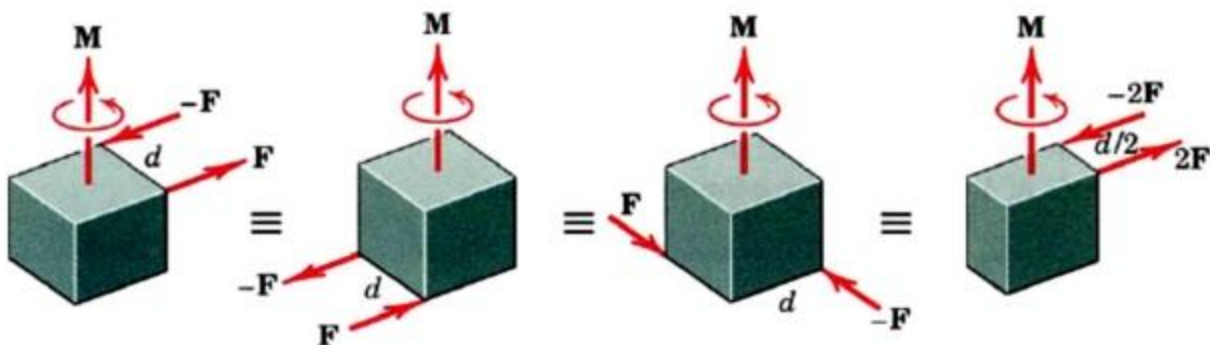
or

$$M = Fd$$



Its direction is counterclockwise when viewed from above for the case illustrated. Note especially that the magnitude of the couple is independent of the distance a which locates the forces with respect to the moment center O . It follows that the moment of a couple has the same value for all moment centers.

Changing the values of F and d does not change a given couple as long as the product Fd remains the same. Likewise, a couple is not affected in the force act in a different but parallel plane. The figure below shows four different configurations of the same couple M . In each of the four cases, the couples are equivalent and are described by the same free vector which represents the identical tendencies to rotate the bodies.



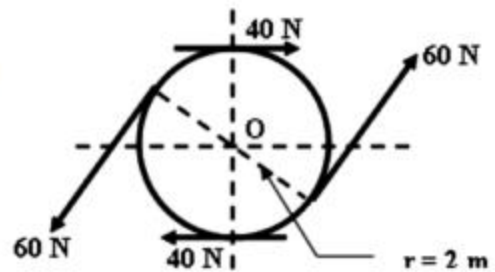
Ex (1)

Compute the magnitude and direction of the resultant couples action on the body shown

Solution :

$$M_c = 60 * 4 - 40 * 4$$

$$= 240 - 160 = 80 \text{ N} \cdot \text{m}$$

**Ex (2)**

A lug wrench is used to tighten a hex-head bolt , Determine the magnitude (F) of the equal forces exerted on the six contact points as shown in fig.

Solution

On the lug wrench :

$$M_c = F * d$$

$$= 250 * 350 * 2$$

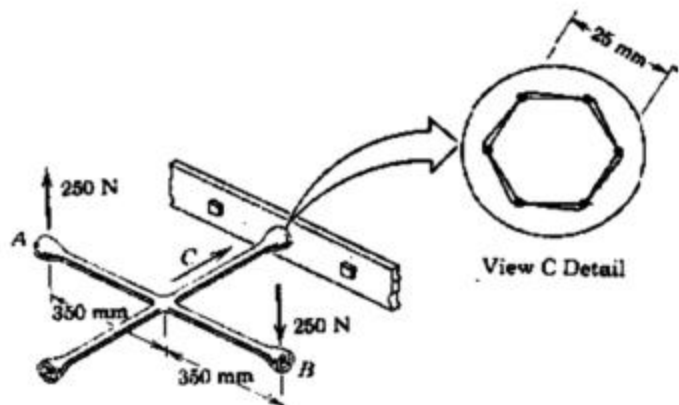
$$= 175000 \text{ N} \cdot \text{mm}$$

On the nut :

$$M_c = 3 F * d$$

$$175000 = 3 F * 25$$

$$F = 175000 / 75 = 2333.33 \text{ N}$$

**Ex (3):**

The rigid structural member is subjected to a couple consisting of the two 100-N forces. Replace this couple by an equivalent couple consisting of the two forces P and -P, each of which has a magnitude of 400 N. Determine the proper angle θ .

Solution:

The original couple is counterclockwise when the plane of the forces is viewed from above, and its magnitude is:

$$[M = Fd] \quad M = 100(0.1) = 10 \text{ N} \cdot \text{m}$$

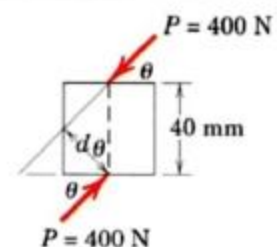
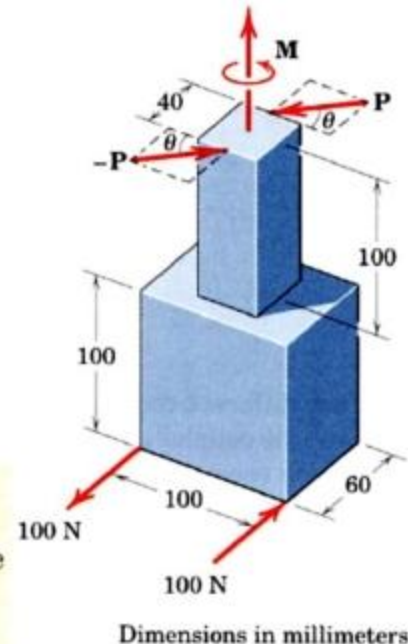
The forces **P** and **-P** produce a counterclockwise couple

$$M = 400(0.040) \cos \theta$$

Equating the two expressions gives

$$10 = 400(0.040) \cos \theta$$

$$\theta = \cos^{-1} \frac{10}{16} = 51.3^\circ$$



Ex (4):

Replace the horizontal 80-lb force acting on the lever by an equivalent system consisting of a force at O and a couple.

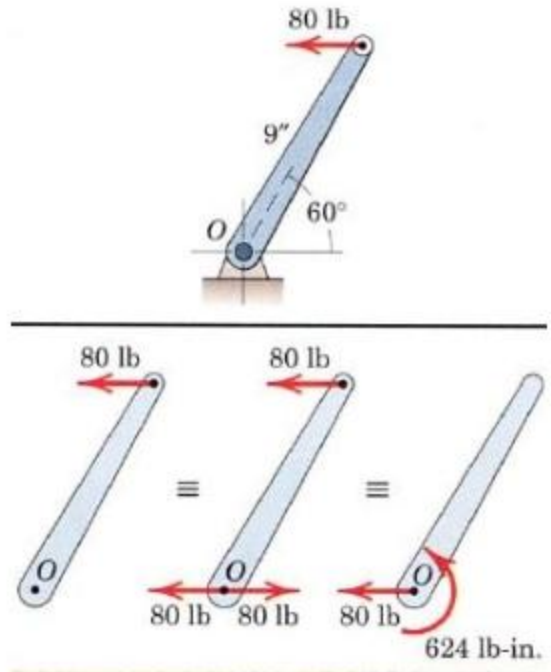
Solution:

We apply two equal and opposite 80-lb forces at O and identify the counterclockwise couple

$$[M=F.d]$$

$$M=80 (90 \sin 60^\circ)= 624 \text{ lb-in.}$$

Thus, the original force is equivalent to the 80-lb force at O and the 624 lb-in couple as shown in the third of the three equivalent figures.

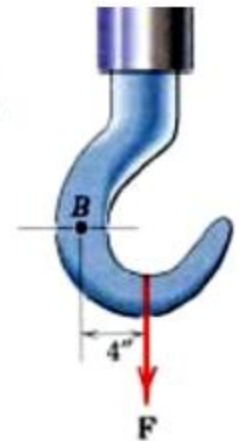


Ex (5):

In the design of lifting hook , the action of the applied force (F) at the critical section of the hook is a direct pull at (B) and a couple . if the magnitude of the couple is (4000 lb.ft) , Determine the magnitude of the force (F) .

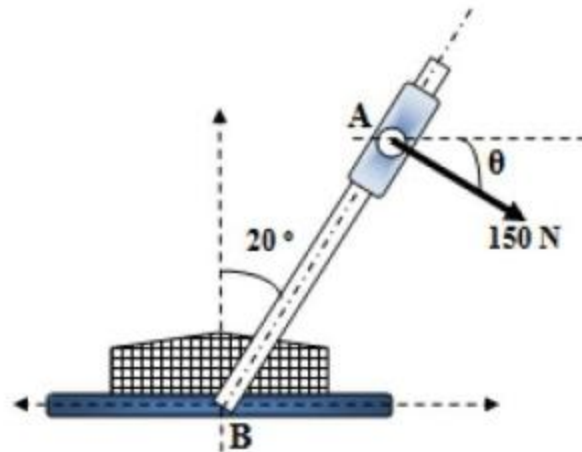
Solution

$$\begin{aligned} M_c &= F * d \\ F &= M_c / d \\ &= 4000 * 12 / 4 \\ &= 12000 \text{ lb} \end{aligned}$$

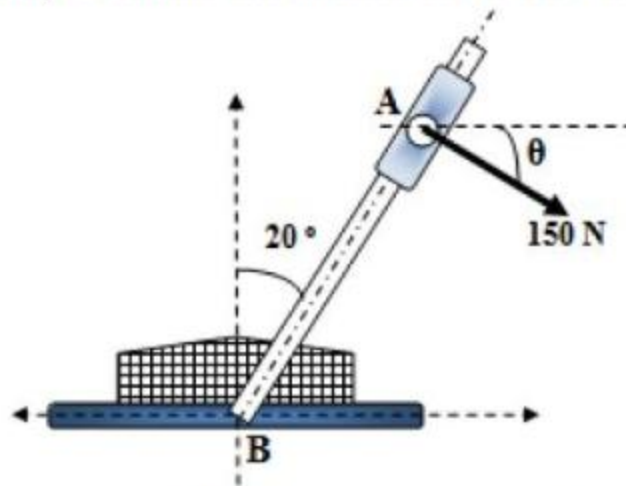


Problems 2:

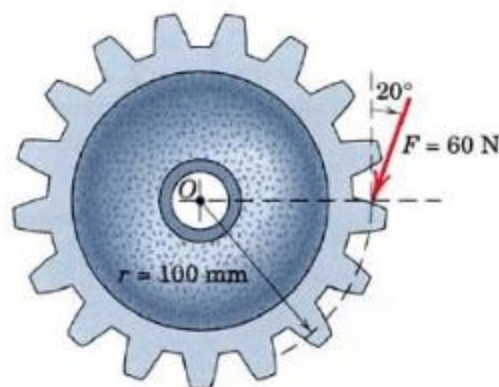
Q2.1: Knowing that the distance AB is 250 mm determine the maximum moment about (B) which can be caused by the (150 N) force . In what direction should the force act ?



Q2.2: A (150 N) force is applied to the control lever at (A) , knowing that the distance AB is (250 mm) Determine the moment of the force about (B) when θ is 50°



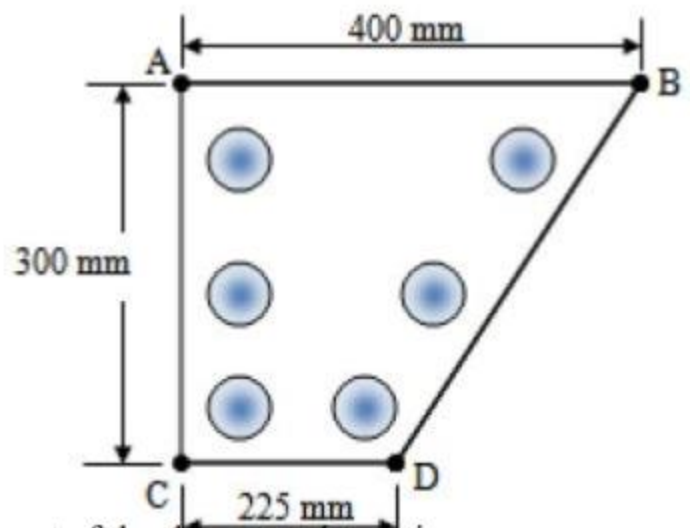
Q2.3: A force F of magnitude 60 N is applied to the gear. Determine the moment of F about point O .



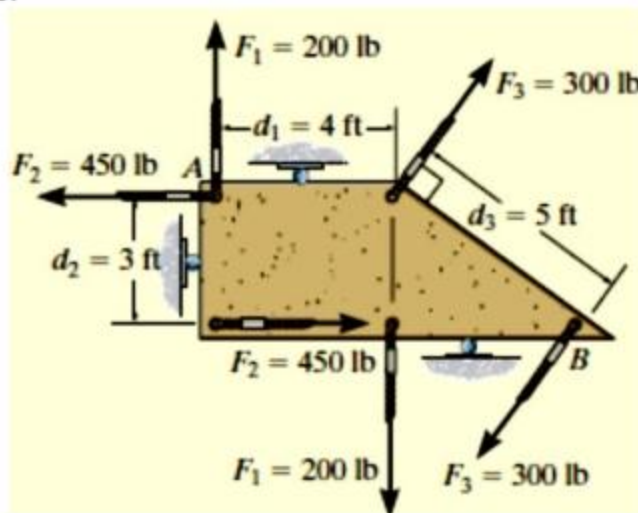
Q2.4:

A multiple – drilling is used to drill simultaneously six holes in the steel plate shown in fig. . Each drill exerts clockwise couple of magnitude (5 N.m) on the plate . Determine an equivalent couple formed by the smallest possible forces acting :

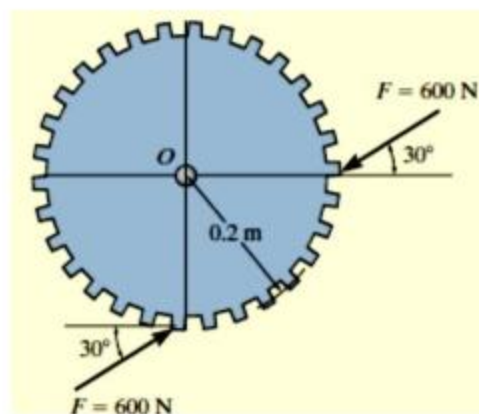
- a - at A and C
- b - at A and D
- c – on the plate



Q2.5: Determine the resultant couple moment of the three couples acting on the plate in Figure.



Q2.6: Determine the magnitude and direction of the couple moment acting on the gear in Figure.



Resultant of non-concurrent coplanar forces system

We will find out the resultant force for many non – concurrent forces acting on a rigid body by using the following equations :

$$R_x = F_1.\cos\theta_1 \mp F_2.\cos\theta_2 \mp F_3.\cos\theta_3 \mp \dots \mp F_n.\cos\theta_n$$

$$R_y = F_1.\sin\theta_1 \mp F_2.\sin\theta_2 \mp F_3.\sin\theta_3 \mp \dots \mp F_n.\sin\theta_n$$

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

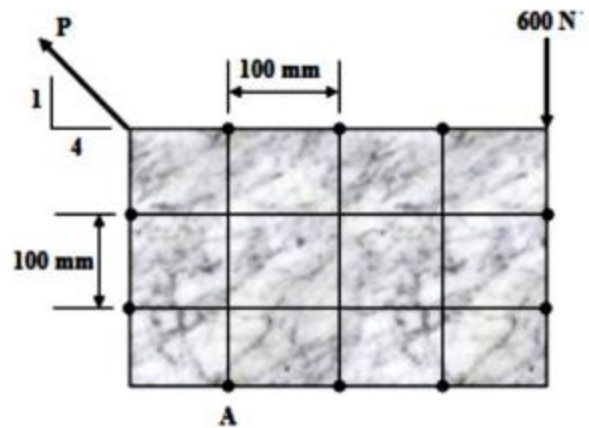
The direction of resultant force may be determined as :

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

~~~~~

#### Ex ( 1 ) :

Determine the force ( P ) shown in fig. knowing that the resultant of the two forces pass through the point ( A ).



#### Solution

$$R_x = F_1.\cos\theta_1 \mp F_2.\cos\theta_2$$

$$R * \cos\theta = -P * \frac{4}{\sqrt{17}}$$

$$R * \cos\theta = -0.9701 * P \dots\dots\dots (1)$$

$$R_y = F_1.\sin\theta_1 \mp F_2.\sin\theta_2$$

$$R * \sin\theta = -600 + P * \frac{1}{\sqrt{17}}$$

$$R * \sin\theta = -600 + 0.2425 P \dots\dots\dots (2)$$

$$\sum M_A = 0$$

$$600 * 300 + P * \frac{1}{\sqrt{17}} * 100 - P * \frac{4}{\sqrt{17}} * 300 = 0$$

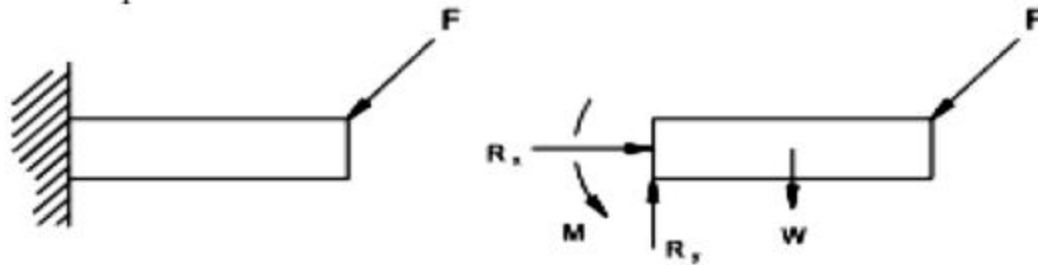
$$180000 + 24.25 P - 291 P = 0$$

$$180000 = 266.7 P$$

$$P = \frac{180000}{266.7} = -674.68 N$$

## FREE BODY DIAGRAM

**Free body diagram:** is a sketch to show all the forces and reactions acting on the body. For example:


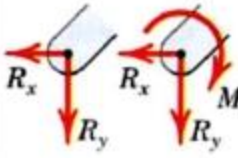

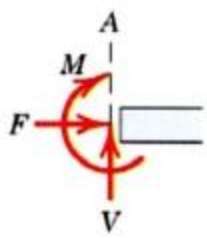

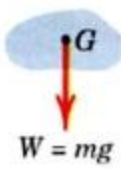
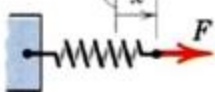
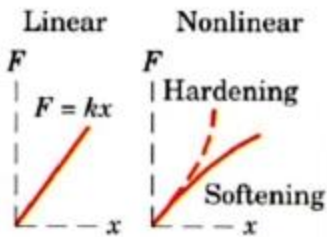



**Free Body Diagram**

The free body diagram includes external forces applied to the body and external reaction forces resulting from the method of supporting the body.

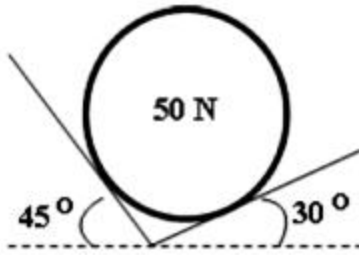
### Free – body diagram and the mechanical effects

| MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS                                                              |                                                                                                                                                                                                      |
|------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Type of Contact and Force Origin                                                                                       | Action on Body to Be Isolated                                                                                                                                                                        |
| <p>1. Flexible cable, belt, chain, or rope</p> <p>Weight of cable negligible</p> <p>Weight of cable not negligible</p> | <p>Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.</p>                                                                                       |
| <p>2. Smooth surfaces</p>                                                                                              | <p>Contact force is compressive and is normal to the surface.</p>                                                                                                                                    |
| <p>3. Rough surfaces</p>                                                                                               | <p>Rough surfaces are capable of supporting a tangential component <math>F</math> (frictional force) as well as a normal component <math>N</math> of the resultant contact force <math>R</math>.</p> |
| <p>4. Roller support</p>                                                                                               | <p>Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.</p>                                                                                               |
| <p>5. Freely sliding guide</p>                                                                                         | <p>Collar or slider free to move along smooth guides; can support force normal to guide only.</p>                                                                                                    |

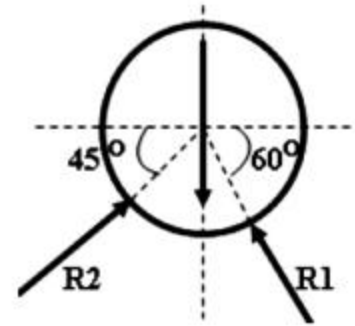
| MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (cont.)                                                                                                                                                                                                               |                                                                                                                                                                                                                                                                                                                                                                                                       |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Type of Contact and Force Origin                                                                                                                                                                                                                                                | Action on Body to Be Isolated                                                                                                                                                                                                                                                                                                                                                                         |
| <p>6. Pin connection</p>                                                                                                                                                                       | <p>Pin free to turn      Pin not free to turn</p>  <p>A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the axis; usually shown as two components <math>R_x</math> and <math>R_y</math>. A pin not free to turn may also support a couple <math>M</math>.</p> |
| <p>7. Built-in or fixed support</p>                                                                                                                                                            |  <p>A built-in or fixed support is capable of supporting an axial force <math>F</math>, a transverse force <math>V</math> (shear force), and a couple <math>M</math> (bending moment) to prevent rotation.</p>                                                                                                      |
| <p>8. Gravitational attraction</p>                                                                                                                                                            |  <p>The resultant of gravitational attraction on all elements of a body of mass <math>m</math> is the weight <math>W = mg</math> and acts toward the center of the earth through the center mass <math>G</math>.</p>                                                                                                |
| <p>9. Spring action</p> <p>Neutral position</p>  <p>Linear      Nonlinear</p> <p><math>F = kx</math></p>  |  <p>Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness <math>k</math> is the force required to deform the spring a unit distance.</p>                                                                                                       |

**Ex (1):**

Draw Free – body diagram for the 50 N sphere shown in fig.

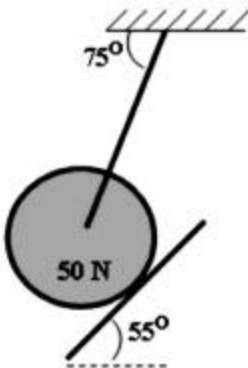


**Solution**

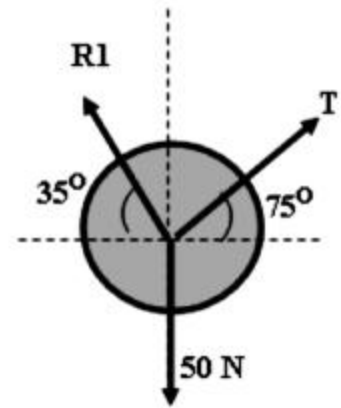


**Ex (2):**

Draw Free – body diagram for the 50 N sphere shown in fig.

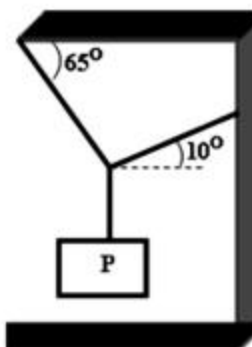


**Solution**

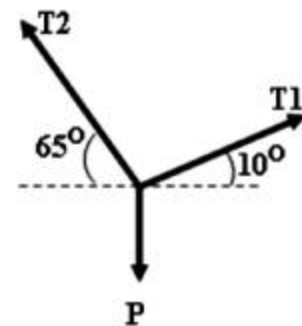


**Ex (3):**

Draw Free – body diagram for the ropes system shown in fig.

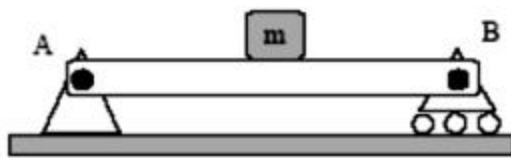


**Solution**



## Examples

### Mass at mid-point on beam (length L)



x-component forces  
 $F_{Ax} = 0$

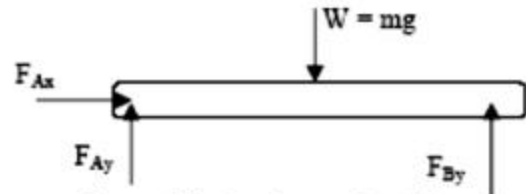
y-component forces  
 $F_{Ay} + F_{By} - W = 0$

moments about mid-point (or use A or B)  
 $-\frac{1}{2}L F_{Ay} + \frac{1}{2}L F_{By} = 0$

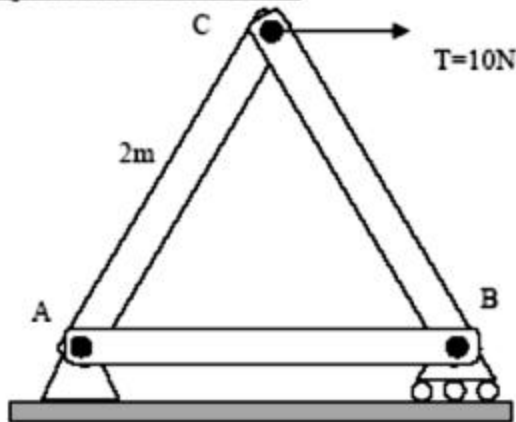
Final result

$F_{Ax} = 0, \quad F_{Ay} = F_{By} = \frac{1}{2} W = \frac{1}{2} mg$

### Free body diagram



### Simple structure with a cable



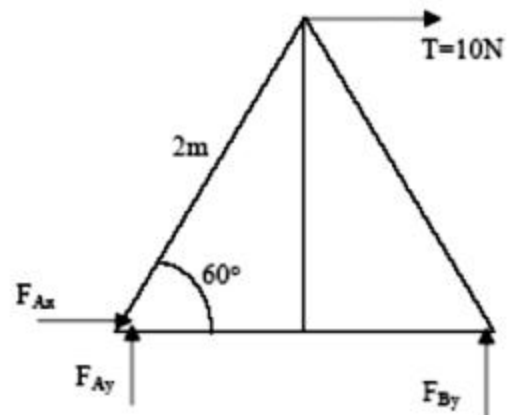
x-component forces  
 $F_{Ax} + T = 0$

y-component forces  
 $F_{Ay} + F_{By} = 0$

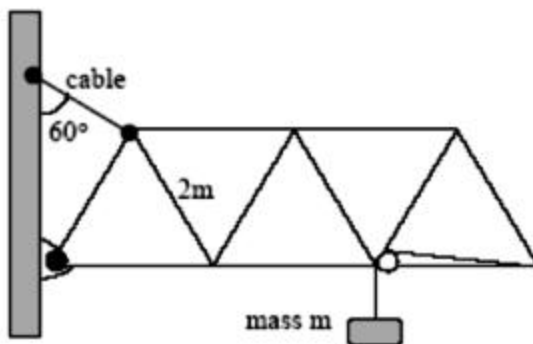
moments about A  
 $2 F_{By} - 2 \sin 60^\circ T = 0$

Final result

$F_{Ax} = -10 \text{ N}, \quad F_{Ay} = -F_{By} = -10 \sin 60^\circ = -8.66 \text{ N}$



### Structure with cable and mass



moments about A  
 $2T - 4W = 0$

x-component forces  
 $F_{Ax} - T \sin 60^\circ = 0$

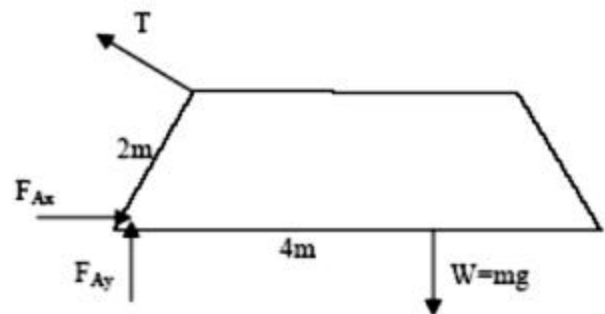
y-component forces  
 $F_{Ay} + T \cos 60^\circ - W = 0$

Final result

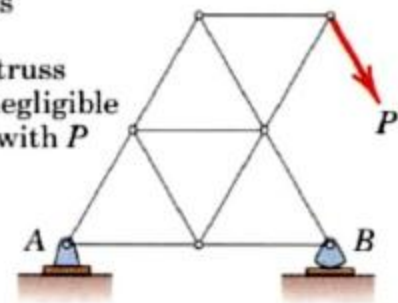
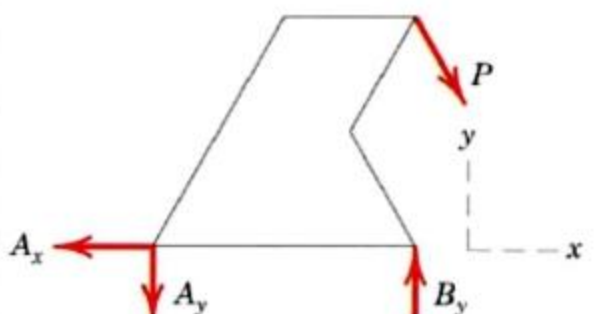
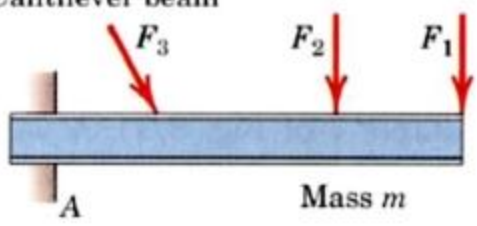
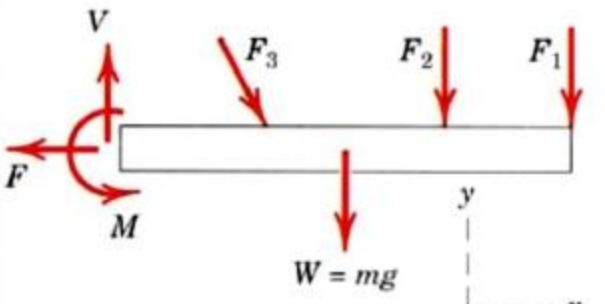
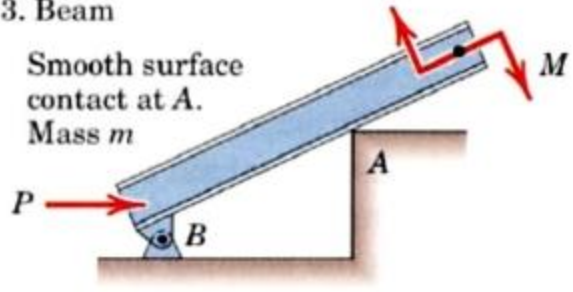
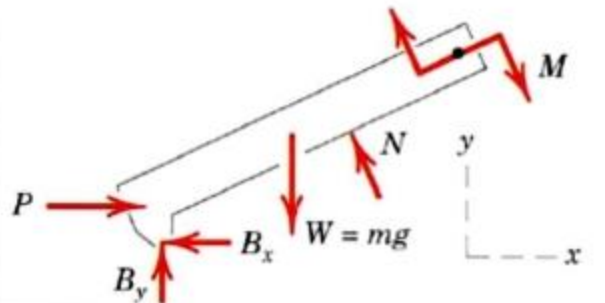
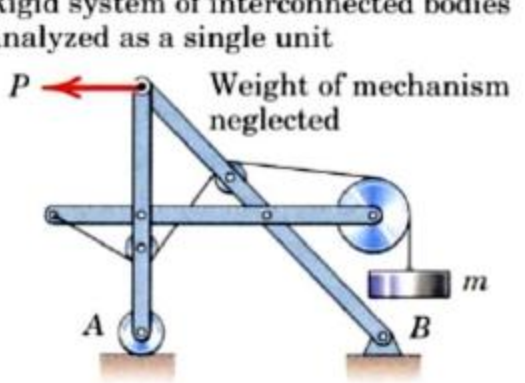
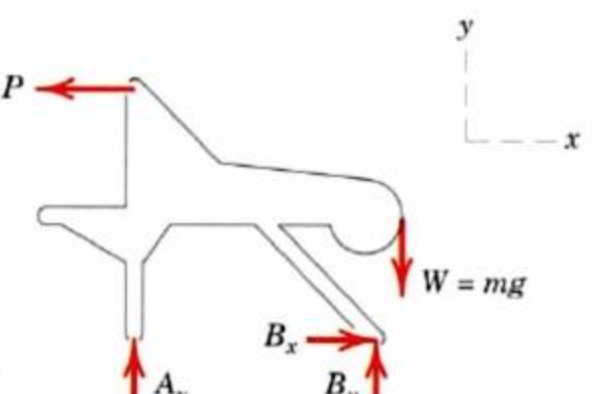
$T = 2W = 2mg$

$F_{Ax} = T \sin 60^\circ = 1.73 mg$

$F_{Ay} = W - T \cos 60^\circ = 0$



**SAMPLE FREE-BODY DIAGRAMS**

| Mechanical System                                                                                                                                                                                  | Free-Body Diagram of Isolated Body                                                   |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|
| <p>1. Plane truss</p> <p>Weight of truss assumed negligible compared with <math>P</math></p>                      |    |
| <p>2. Cantilever beam</p>                                                                                         |    |
| <p>3. Beam</p> <p>Smooth surface contact at A.</p> <p>Mass <math>m</math></p>                                    |   |
| <p>4. Rigid system of interconnected bodies analyzed as a single unit</p> <p>Weight of mechanism neglected</p>  |  |

#### 4. Equilibrium of Forces Systems.

1- For Coplanar forces system :

a- concurrent coplanar forces system

$$\mathbf{R_x = 0 , R_y = 0 , R = 0}$$

b - non-concurrent coplanar forces system

$$\mathbf{R_x = 0 , R_y = 0 , R = 0 , \Sigma M = 0}$$

2 - Non coplanar forces system :

a. concurrent non-coplanar forces system

$$\mathbf{R_x = 0 , R_y = 0 , R = 0 , \Sigma M = 0}$$

b. non-concurrent non-coplanar forces system

$$\mathbf{R_x = 0 , R_y = 0 , R = 0 , \Sigma M = 0}$$

#### Ex ( 1 ) :

Determine the tension in the cord and the reaction of inclined plane acting on the sphere of ( 50 N ) weight shown in fig.

#### Solution:

Draw **F.B.D** for the sphere , then :

$$\Sigma F_x = 0$$

$$T \cos 75 - R_1 \cos 35 = 0 \dots\dots\dots ( 1 )$$

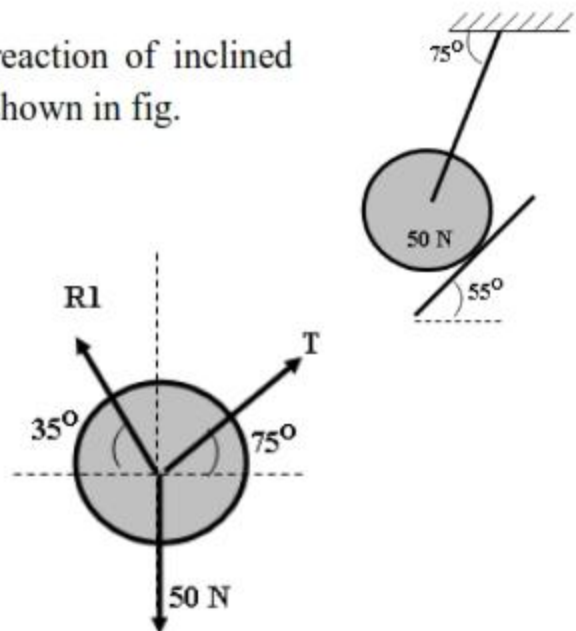
$$\Sigma F_y = 0$$

$$T \sin 75 + R_1 \sin 35 - 50 = 0 \dots\dots\dots( 2 )$$

Subst. ( 2 ) in ( 1 ) we get :

$$T = 4361 \text{ N}$$

$$R_1 = 137.7 \text{ N}$$



#### Ex ( 2 ) :

Determine the reactions at the points (A) and (B) , the angle beam was in equilibrium state as shown in fig .

#### Solution

$$\Sigma M ( A ) = 0$$

$$500 * 10 - N * 15 = 0$$

$$N = 5000 / 15 = 333.34 \text{ N}$$

$$\Sigma F_y = 0$$

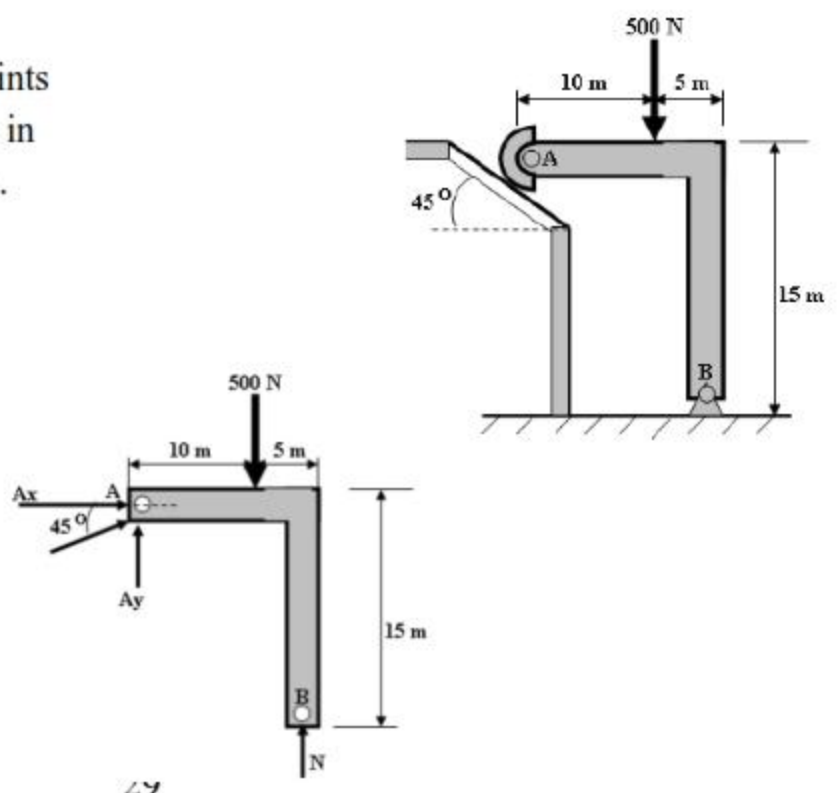
$$A_y + N - 500 = 0$$

$$A_y + 333.34 - 500 = 0$$

$$A_y = 166.67 \text{ N}$$

$$\Sigma F_x = 0 , A_x = 0$$

$$R_A = A_y = 166.67 \text{ N}$$



**Ex (3):**

Determine the tension forces (**T1**) and (**T2**) in the equilibrium system shown in fig.

**Solution**

$\Sigma F_x = 0$

$T1 \cdot \cos ( 0 ) + T2 \cdot \cos ( 60 ) - 2000 \cos ( 15 ) = 0$

$T1 + 0.5 T2 - 1931.85 = 0 \dots\dots\dots ( 1 )$

$\Sigma F_y = 0$

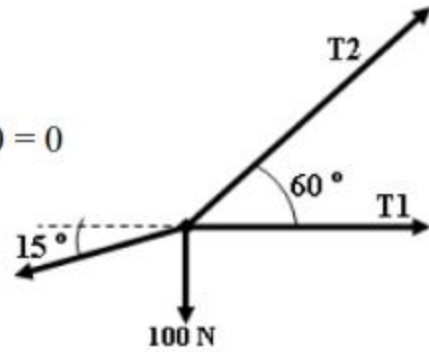
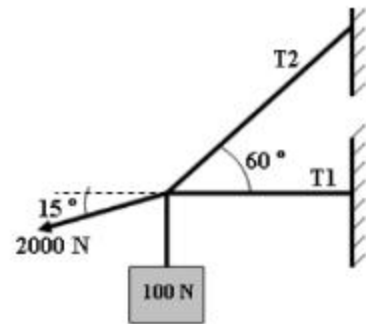
$T1 \cdot \sin ( 0 ) + T2 \cdot \sin ( 60 ) - 2000 \cdot \sin ( 15 ) - 100 = 0$

$0.866 T2 - 617.63 = 0$

$T2 = 713.2 \text{ N}$

Subs. in ( 1 )

$T1 = 1575 \text{ N}$



**Ex (4):**

Determine the tension in each cord shown in fig. ( **TA** , **TB** , **TC** , **TD** ).

**Solution**

By using Lami' s rule :

$\frac{T_D}{\sin 150} = \frac{40}{\sin 90}$

$\frac{T_D}{0.5} = \frac{40}{1}$

$T_D = 20 \text{ N}$

$\frac{T_C}{\sin 120} = \frac{40}{\sin 90}$

$\frac{T_C}{0.866} = \frac{40}{1}$

$T_C = 34.64 \text{ N}$

$\frac{T_B}{\sin 90} = \frac{T_C}{\sin 135}$

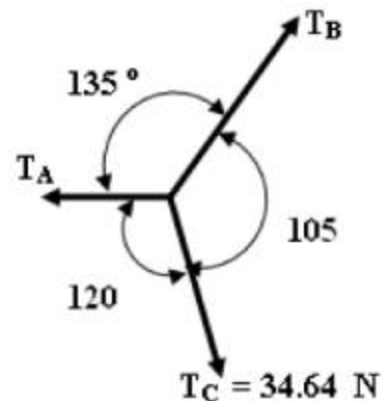
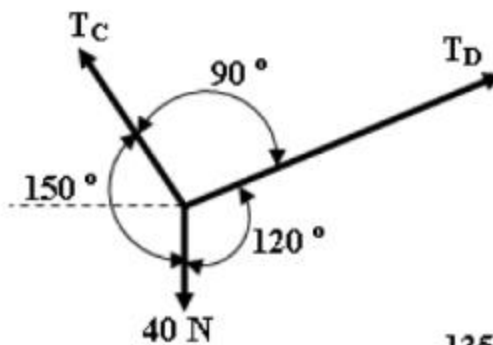
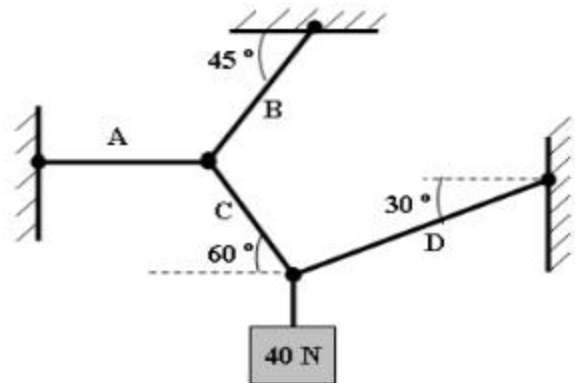
$\frac{T_B}{0.5} = \frac{34.64}{0.707}$

$T_B = 48.98 \text{ N}$

$\frac{T_A}{\sin 135} = \frac{T_C}{\sin 135}$

$\frac{T_A}{0.707} = \frac{34.64}{0.707}$

$T_A = 34.64 \text{ N}$



**Ex ( 5 ) :**

Find out the reaction on the cylinder ( A )  
and the total force acting on the pin ( O )

**Solution**

$$\Sigma M ( O ) = 0$$

$$2 * 250 - R_A * 400 = 0$$

$$400 R_A = 500$$

$$R_A = 1.25 \text{ KN}$$

$$\Sigma F_y = 0$$

$$O_y - 2 = 0$$

$$O_y = 2 \text{ KN}$$

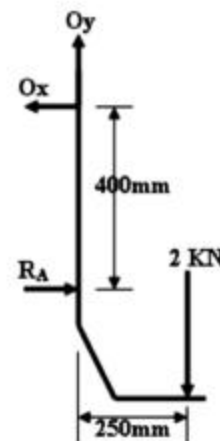
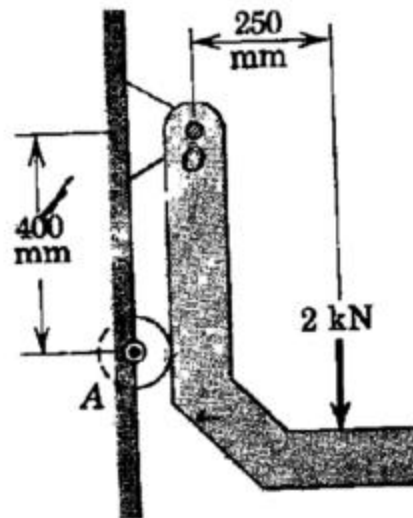
$$\Sigma F_x = 0$$

$$R_A - O_x = 0$$

$$O_x = R_A = 1.25 \text{ KN}$$

$$F = \sqrt{(O_x)^2 + (O_y)^2}$$

$$F = \sqrt{(1.25)^2 + (2)^2} = 2.35 \text{ N}$$



F.B.D

**Ex ( 6 ) :**

Find out the reactions at the points ( A ) & ( B ) .

**Solution**

$$\Sigma M ( B ) = 0$$

$$- R_A * 600 + 300 * 9.8 * 1000 = 0$$

$$- 600 R_A + 2940 000 = 0$$

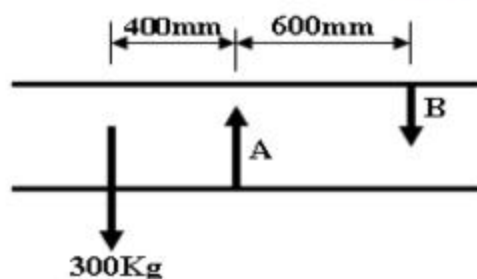
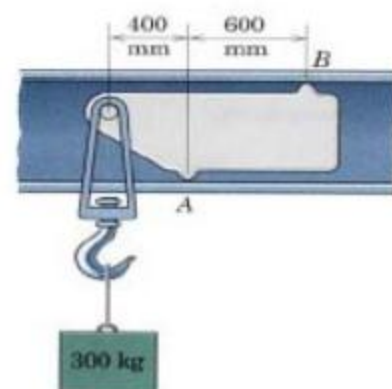
$$R_A = 4900 \text{ N} = 4.9 \text{ KN}$$

$$\Sigma F_y = 0$$

$$R_A - R_B - 300 * 9.8 = 0$$

$$4900 - R_B - 2943 = 0$$

$$R_B = 1950 \text{ N} = 1.95 \text{ KN}$$



## 5. Friction:

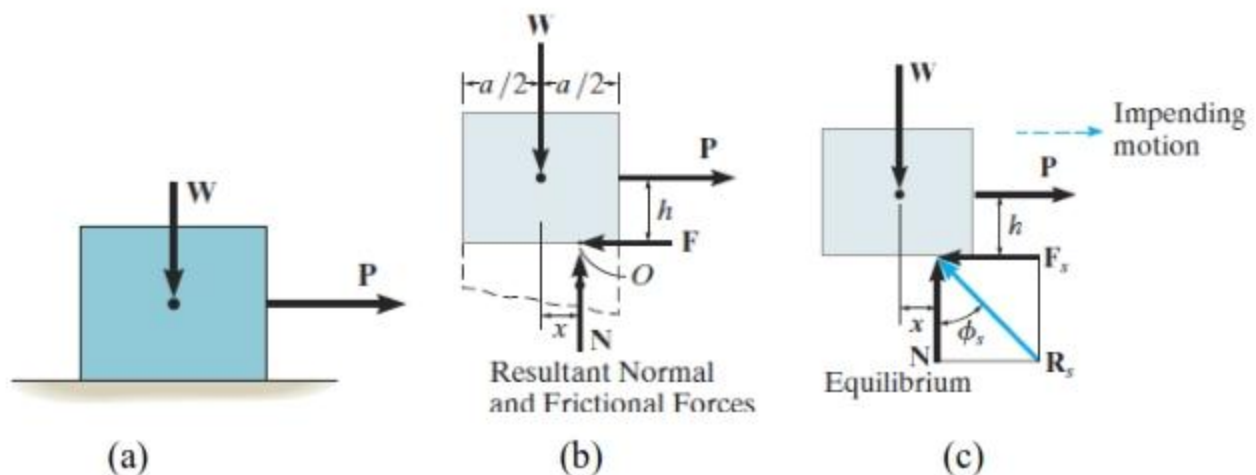
Friction is a force that resists the movement of two contacting surfaces that slide relative to one another.

Considering the effects caused by pulling horizontally force  $P$  on a block of uniform weight  $W$  which is resting on a rough horizontal surface. From equilibrium, two reaction forces generated;

Normal force  $N$  to resist the weight  $W$  and

Friction force  $F$  to resist the pulling force  $P$ .

Notice that  $N$  acts a distance  $x$  to the right of the line of action of  $W$ , Fig. (b) which coincides with geometric center of in order to balance the effect caused by  $P$ .



**Impending Motion:** If the block will tend to slip, the maximum value called the limiting static frictional force, Fig. (c), which is directly proportional to the resultant normal force  $N$ . Expressed mathematically:

$$F_s = \mu_s N$$

- Where:  $\mu_s$  (mu "sub" s), is called the coefficient of static friction. Typical values for  $\mu_s$  are given in Table 8-1.
- When the block is on verge of sliding, the normal force  $N$  and frictional force  $F_s$  create a resultant  $R_s$  Fig. (c).

The angle  $\phi_s$  (phi "sub" s) that makes with  $N$  is called the angle of static friction.

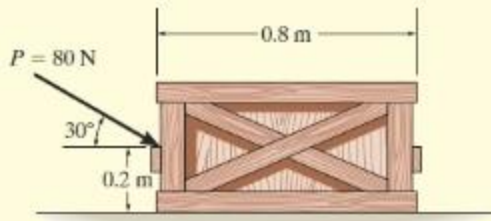
$$\phi_s = \tan^{-1}\left(\frac{F_s}{N}\right) = \tan^{-1}\left(\frac{\mu_s N}{N}\right) = \tan^{-1} \mu_s$$

**Table 8-1**  
Typical Values for  $\mu_s$

| Contact Materials    | Coefficient of Static Friction ( $\mu_s$ ) |
|----------------------|--------------------------------------------|
| Metal on ice         | 0.03–0.05                                  |
| Wood on wood         | 0.30–0.70                                  |
| Leather on wood      | 0.20–0.50                                  |
| Leather on metal     | 0.30–0.60                                  |
| Aluminum on aluminum | 1.10–1.70                                  |

### Ex (1):

The uniform crate shown in Fig. 8–7a has a mass of 20 kg. If a force  $P = 80 \text{ N}$  is applied to the crate, determine if it remains in equilibrium. The coefficient of static friction is  $\mu_s = 0.3$ .



(a)

Fig. 8–7

#### SOLUTION

**Free-Body Diagram.** As shown in Fig. 8–7b, the *resultant* normal force  $N_C$  must act a distance  $x$  from the crate's center line in order to counteract the tipping effect caused by  $\mathbf{P}$ . There are *three unknowns*,  $F$ ,  $N_C$ , and  $x$ , which can be determined strictly from the *three* equations of equilibrium.

#### Equations of Equilibrium.

$$\rightarrow \sum F_x = 0; \quad 80 \cos 30^\circ \text{ N} - F = 0$$

$$+\uparrow \sum F_y = 0; \quad -80 \sin 30^\circ \text{ N} + N_C - 196.2 \text{ N} = 0$$

$$\zeta + \sum M_O = 0; \quad 80 \sin 30^\circ \text{ N}(0.4 \text{ m}) - 80 \cos 30^\circ \text{ N}(0.2 \text{ m}) + N_C(x) = 0$$

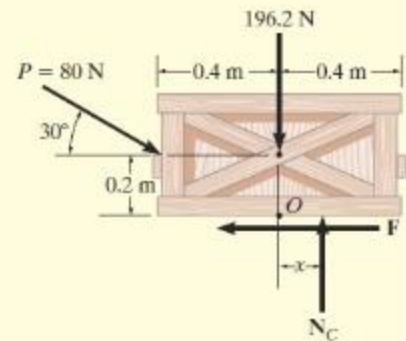
Solving,

$$F = 69.3 \text{ N}$$

$$N_C = 236 \text{ N}$$

$$x = -0.00908 \text{ m} = -9.08 \text{ mm}$$

Since  $x$  is negative it indicates the *resultant* normal force acts (slightly) to the *left* of the crate's center line. No tipping will occur since  $x < 0.4 \text{ m}$ . Also, the *maximum* frictional force which can be developed at the surface of contact is  $F_{\max} = \mu_s N_C = 0.3(236 \text{ N}) = 70.8 \text{ N}$ . Since  $F = 69.3 \text{ N} < 70.8 \text{ N}$ , the crate will *not slip*, although it is very close to doing so.



(b)

**Ex (2):**

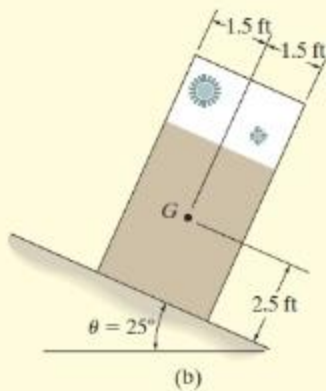


(a)

It is observed that when the bed of the dump truck is raised to an angle of  $\theta = 25^\circ$  the vending machines will begin to slide off the bed, Fig. 8-8a. Determine the static coefficient of friction between a vending machine and the surface of the truckbed.

**SOLUTION**

An idealized model of a vending machine resting on the truckbed is shown in Fig. 8-8b. The dimensions have been measured and the center of gravity has been located. We will assume that the vending machine weighs  $W$ .



(b)

**Free-Body Diagram.** As shown in Fig. 8-8c, the dimension  $x$  is used to locate the position of the resultant normal force  $N$ . There are four unknowns,  $N$ ,  $F$ ,  $\mu_s$ , and  $x$ .

**Equations of Equilibrium.**

$$+\searrow \Sigma F_x = 0; \quad W \sin 25^\circ - F = 0 \quad (1)$$

$$+\nearrow \Sigma F_y = 0; \quad N - W \cos 25^\circ = 0 \quad (2)$$

$$\zeta + \Sigma M_O = 0; \quad -W \sin 25^\circ(2.5 \text{ ft}) + W \cos 25^\circ(x) = 0 \quad (3)$$

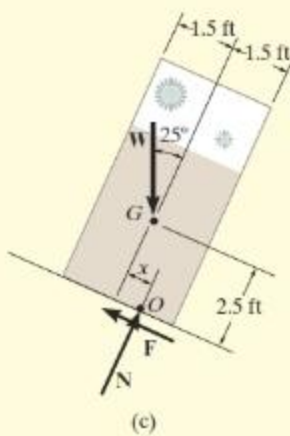
Since slipping impends at  $\theta = 25^\circ$ , using Eqs. 1 and 2, we have

$$F_s = \mu_s N; \quad W \sin 25^\circ = \mu_s (W \cos 25^\circ)$$

$$\mu_s = \tan 25^\circ = 0.466 \quad \text{Ans.}$$

The angle of  $\theta = 25^\circ$  is referred to as the *angle of repose*, and by comparison, it is equal to the angle of static friction,  $\theta = \phi_s$ . Notice from the calculation that  $\theta$  is independent of the weight of the vending machine, and so knowing  $\theta$  provides a convenient method for determining the coefficient of static friction.

**NOTE:** From Eq. 3, we find  $x = 1.17 \text{ ft}$ . Since  $1.17 \text{ ft} < 1.5 \text{ ft}$ , indeed the vending machine will slip before it can tip as observed in Fig. 8-8a.



(c)

**Fig. 8-8**

**Ex (3):**

The uniform 10-kg ladder in Fig. 8–9*a* rests against the smooth wall at *B*, and the end *A* rests on the rough horizontal plane for which the coefficient of static friction is  $\mu_s = 0.3$ . Determine the angle of inclination  $\theta$  of the ladder and the normal reaction at *B* if the ladder is on the verge of slipping.

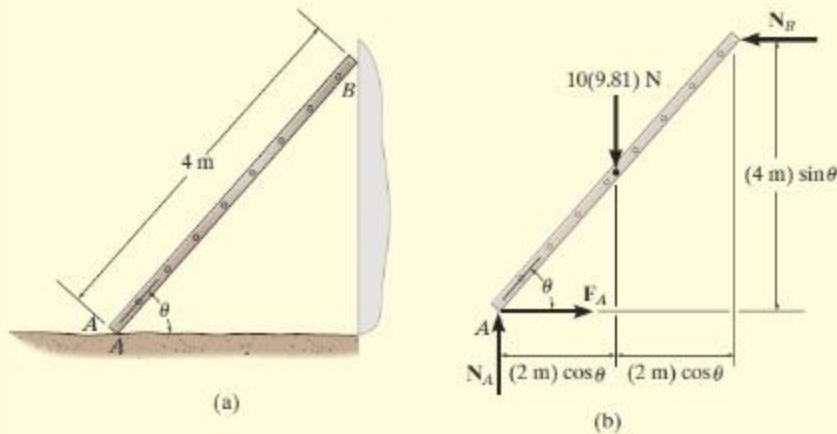


Fig. 8–9

**SOLUTION**

**Free-Body Diagram.** As shown on the free-body diagram, Fig. 8–9*b*, the frictional force  $F_A$  must act to the right since impending motion at *A* is to the left.

**Equations of Equilibrium and Friction.** Since the ladder is on the verge of slipping, then  $F_A = \mu_s N_A = 0.3N_A$ . By inspection,  $N_A$  can be obtained directly.

$$+\uparrow \Sigma F_y = 0; \quad N_A - 10(9.81) \text{ N} = 0 \quad N_A = 98.1 \text{ N}$$

Using this result,  $F_A = 0.3(98.1 \text{ N}) = 29.43 \text{ N}$ . Now  $N_B$  can be found.

$$\begin{aligned} \rightarrow \Sigma F_x = 0; \quad 29.43 \text{ N} - N_B &= 0 \\ N_B &= 29.43 \text{ N} = 29.4 \text{ N} \quad \text{Ans.} \end{aligned}$$

Finally, the angle  $\theta$  can be determined by summing moments about point *A*.

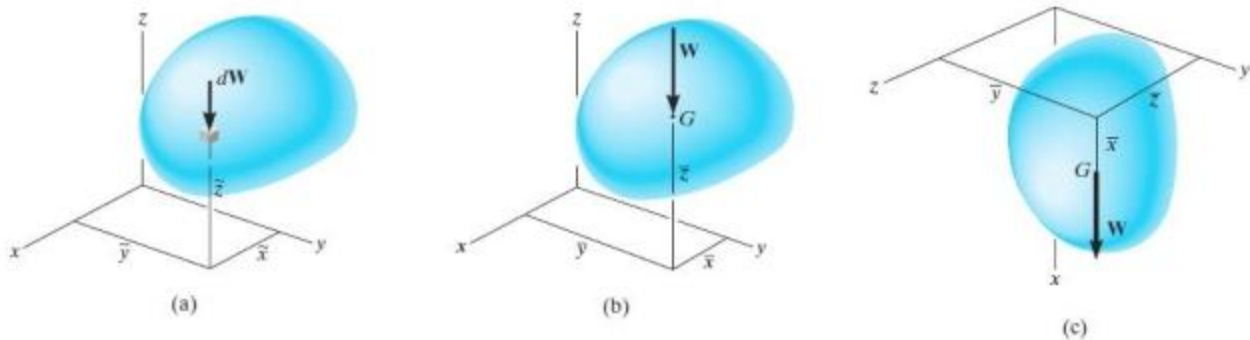
$$\begin{aligned} \zeta + \Sigma M_A = 0; \quad (29.43 \text{ N})(4 \text{ m}) \sin \theta - [10(9.81) \text{ N}](2 \text{ m}) \cos \theta &= 0 \\ \frac{\sin \theta}{\cos \theta} = \tan \theta &= 1.6667 \\ \theta &= 59.04^\circ = 59.0^\circ \quad \text{Ans.} \end{aligned}$$

## 6. Center of Mass and Centroid:

Previously, we treat all forces as a concentrated along their line of action and at their point of application. Actually concentrated force does not exist in the exact sense, since every external force applied mechanically to a body is distributed over a finite contact area, however small. The force applied by tire are distributed on contact area, when analyzing the force the contact area is negligible and replaced by resultant at the center of the area.



A body is composed of an infinite number of particles of differential size, and so if the body is located within a gravitational field, then each of these particles will have a weight  $dW$ . These weights will form an approximately parallel force system, and the resultant of this system is the total weight of the body, which passes through a single point called the center of gravity,  $G$  as shown in Figure below.



**Figure 9.1**

The weight of the body is the sum of the weights of all of its particles, that is

$$+\downarrow F_R = \Sigma F_z; \quad W = \int dW$$

The location of the center of gravity, measured from the  $y$  axis, is determined by equating the moment of  $W$  about the  $y$  axis, Fig. b, to the sum of the moments of the weights of the particles about this same axis. Similarly, if moments are summed about the  $x$  and  $z$  axes,

$$(M_R)_y = \Sigma M_y; \quad \bar{x}W = \int \tilde{x}dW$$

$$(M_R)_x = \Sigma M_x; \quad \bar{y}W = \int \tilde{y}dW$$

$$(M_R)_z = \Sigma M_z; \quad \bar{z}W = \int \tilde{z}dW$$

Therefore, the location of the center of gravity  $G$  with respect to the  $x$ ,  $y$ , and  $z$  axes becomes;

$$\bar{x} = \frac{\int \tilde{x} dW}{\int dW} \quad \bar{y} = \frac{\int \tilde{y} dW}{\int dW} \quad \bar{z} = \frac{\int \tilde{z} dW}{\int dW}$$

9-1

$\bar{x}$ ,  $\bar{y}$ ,  $\bar{z}$  are the coordinates of the center of gravity  $G$ , Fig. 9-1b.

$\tilde{x}$ ,  $\tilde{y}$ ,  $\tilde{z}$  are the coordinates of each particle in the body, Fig. 9-1a.

**Center of Mass of a Body.** To locate the body's center of mass  $C_m$ , Fig.9-2. This location can be determined by substituting  $dW = g dm$  into Eqs. 9-2. Since  $g$  is constant, it cancels out, and so

$$\bar{x} = \frac{\int \tilde{x} dm}{\int dm} \quad \bar{y} = \frac{\int \tilde{y} dm}{\int dm} \quad \bar{z} = \frac{\int \tilde{z} dm}{\int dm}$$

9-2

**Centroid of a Volume.** If the body in Fig. 9-3 is made from a homogeneous material, then its density  $\rho$  (rho) will be constant. Therefore, a differential element of volume  $dV$  has a mass  $dm = \rho dV$ . Substituting this into Eqs. 9-2 and canceling out, we obtain formulas that locate the centroid  $C$  or geometric center of the body; namely

$$\bar{x} = \frac{\int_V \tilde{x} dV}{\int_V dV} \quad \bar{y} = \frac{\int_V \tilde{y} dV}{\int_V dV} \quad \bar{z} = \frac{\int_V \tilde{z} dV}{\int_V dV}$$

9-3

As well as the **Centroid of an Area** is;

$$\bar{x} = \frac{\int_A \tilde{x} dA}{\int_A dA} \quad \bar{y} = \frac{\int_A \tilde{y} dA}{\int_A dA}$$

9-4

And the **Centroid of a Line** is;

$$\bar{x} = \frac{\int_L \tilde{x} dL}{\int_L dL} \quad \bar{y} = \frac{\int_L \tilde{y} dL}{\int_L dL}$$

9-5

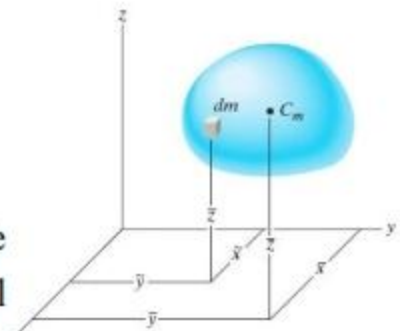


Fig. 9-2

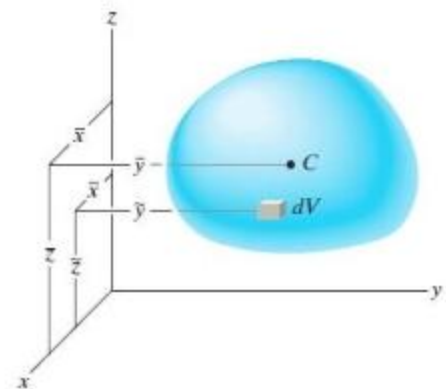


Fig. 9-3

**Ex (1):**

Locate the centroid of the circular wire segment shown in Fig. 9-9.

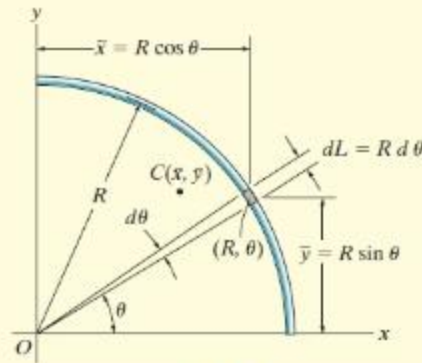


Fig. 9-9

**SOLUTION**

Polar coordinates will be used to solve this problem since the arc is circular.

**Differential Element.** A differential circular arc is selected as shown in the figure. This element intersects the curve at  $(R, \theta)$ .

**Length and Moment Arm.** The length of the differential element is  $dL = R d\theta$ , and its centroid is located at  $\tilde{x} = R \cos \theta$  and  $\tilde{y} = R \sin \theta$ .

**Integrations.** Applying Eqs. 9-5 and integrating with respect to  $\theta$ , we obtain

$$\bar{x} = \frac{\int_L \tilde{x} dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R \cos \theta) R d\theta}{\int_0^{\pi/2} R d\theta} = \frac{R^2 \int_0^{\pi/2} \cos \theta d\theta}{R \int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \text{ Ans.}$$

$$\bar{y} = \frac{\int_L \tilde{y} dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R \sin \theta) R d\theta}{\int_0^{\pi/2} R d\theta} = \frac{R^2 \int_0^{\pi/2} \sin \theta d\theta}{R \int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \text{ Ans.}$$

**NOTE:** As expected, the two coordinates are numerically the same due to the symmetry of the wire.

## Ex (2):

Determine the distance  $\bar{y}$  measured from the  $x$  axis to the centroid of the area of the triangle shown in Fig. 9-10.

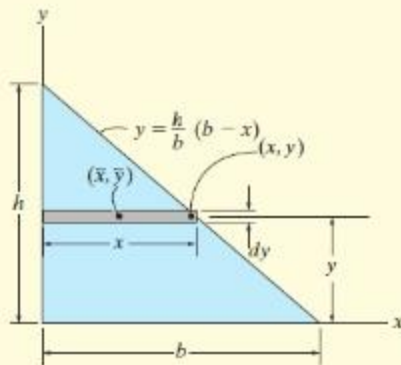


Fig. 9-10

### SOLUTION

**Differential Element.** Consider a rectangular element having a thickness  $dy$ , and located in an arbitrary position so that it intersects the boundary at  $(x, y)$ , Fig. 9-10.

**Area and Moment Arms.** The area of the element is  $dA = x dy = \frac{b}{h}(h-y) dy$ , and its centroid is located a distance  $\tilde{y} = y$  from the  $x$  axis.

**Integration.** Applying the second of Eqs. 9-4 and integrating with respect to  $y$  yields

$$\begin{aligned}\bar{y} &= \frac{\int_A \tilde{y} dA}{\int_A dA} = \frac{\int_0^h y \left[ \frac{b}{h}(h-y) dy \right]}{\int_0^h \frac{b}{h}(h-y) dy} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh} \\ &= \frac{h}{3} \qquad \text{Ans.}\end{aligned}$$

**NOTE:** This result is valid for any shape of triangle. It states that the centroid is located at one-third the height, measured from the base of the triangle.

**Ex (3):**

Locate the centroid for the area of a quarter circle shown in Fig. 9-11.

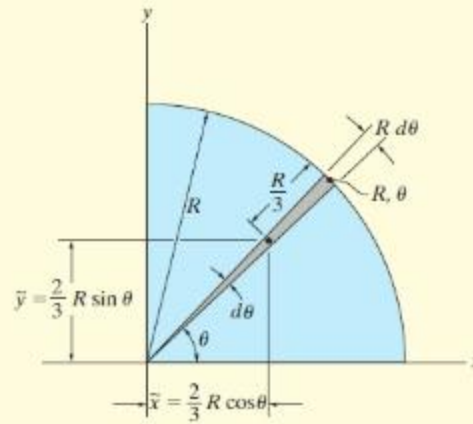


Fig. 9-11

**SOLUTION**

**Differential Element.** Polar coordinates will be used, since the boundary is circular. We choose the element in the shape of a *triangle*, Fig. 9-11. (Actually the shape is a circular sector; however, neglecting higher-order differentials, the element becomes triangular.) The element intersects the curve at point  $(R, \theta)$ .

**Area and Moment Arms.** The area of the element is

$$dA = \frac{1}{2}(R)(R d\theta) = \frac{R^2}{2} d\theta$$

and using the results of Example 9.3, the centroid of the (triangular) element is located at  $\tilde{x} = \frac{2}{3}R \cos \theta$ ,  $\tilde{y} = \frac{2}{3}R \sin \theta$ .

**Integrations.** Applying Eqs. 9-4 and integrating with respect to  $\theta$ , we obtain

$$\bar{x} = \frac{\int_A \tilde{x} dA}{\int_A dA} = \frac{\int_0^{\pi/2} \left(\frac{2}{3}R \cos \theta\right) \frac{R^2}{2} d\theta}{\int_0^{\pi/2} \frac{R^2}{2} d\theta} = \frac{\left(\frac{2}{3}R\right) \int_0^{\pi/2} \cos \theta d\theta}{\int_0^{\pi/2} d\theta} = \frac{4R}{3\pi} \quad \text{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y} dA}{\int_A dA} = \frac{\int_0^{\pi/2} \left(\frac{2}{3}R \sin \theta\right) \frac{R^2}{2} d\theta}{\int_0^{\pi/2} \frac{R^2}{2} d\theta} = \frac{\left(\frac{2}{3}R\right) \int_0^{\pi/2} \sin \theta d\theta}{\int_0^{\pi/2} d\theta} = \frac{4R}{3\pi} \quad \text{Ans.}$$

Ex (4)

Locate the centroid of the area shown in Fig. 9-12a.

**SOLUTION I**

**Differential Element.** A differential element of thickness  $dx$  is shown in Fig. 9-12a. The element intersects the curve at the *arbitrary point*  $(x, y)$ , and so it has a height  $y$ .

**Area and Moment Arms.** The area of the element is  $dA = y dx$ , and its centroid is located at  $\tilde{x} = x, \tilde{y} = y/2$ .

**Integrations.** Applying Eqs. 9-4 and integrating with respect to  $x$  yields

$$\bar{x} = \frac{\int_A \tilde{x} dA}{\int_A dA} = \frac{\int_0^{1\text{m}} xy dx}{\int_0^{1\text{m}} y dx} = \frac{\int_0^{1\text{m}} x^3 dx}{\int_0^{1\text{m}} x^2 dx} = \frac{0.250}{0.333} = 0.75 \text{ m}$$

$$\bar{y} = \frac{\int_A \tilde{y} dA}{\int_A dA} = \frac{\int_0^{1\text{m}} (y/2)y dx}{\int_0^{1\text{m}} y dx} = \frac{\int_0^{1\text{m}} (x^2/2)x^2 dx}{\int_0^{1\text{m}} x^2 dx} = \frac{0.100}{0.333} = 0.3 \text{ m Ans.}$$

**SOLUTION II**

**Differential Element.** The differential element of thickness  $dy$  is shown in Fig. 9-12b. The element intersects the curve at the *arbitrary point*  $(x, y)$ , and so it has a length  $(1 - x)$ .

**Area and Moment Arms.** The area of the element is  $dA = (1 - x) dy$ , and its centroid is located at

$$\tilde{x} = x + \left(\frac{1 - x}{2}\right) = \frac{1 + x}{2}, \tilde{y} = y$$

**Integrations.** Applying Eqs. 9-4 and integrating with respect to  $y$ , we obtain

$$\bar{x} = \frac{\int_A \tilde{x} dA}{\int_A dA} = \frac{\int_0^{1\text{m}} [(1 + x)/2](1 - x) dy}{\int_0^{1\text{m}} (1 - x) dy} = \frac{\frac{1}{2} \int_0^{1\text{m}} (1 - y) dy}{\int_0^{1\text{m}} (1 - \sqrt{y}) dy} = \frac{0.250}{0.333} = 0.75 \text{ m Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y} dA}{\int_A dA} = \frac{\int_0^{1\text{m}} y(1 - x) dy}{\int_0^{1\text{m}} (1 - x) dy} = \frac{\int_0^{1\text{m}} (y - y^{3/2}) dy}{\int_0^{1\text{m}} (1 - \sqrt{y}) dy} = \frac{0.100}{0.333} = 0.3 \text{ m Ans.}$$

**NOTE:** Plot these results and notice that they seem reasonable. Also, for this problem, elements of thickness  $dx$  offer a simpler solution.

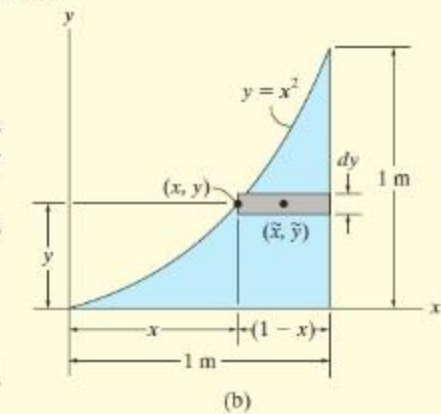
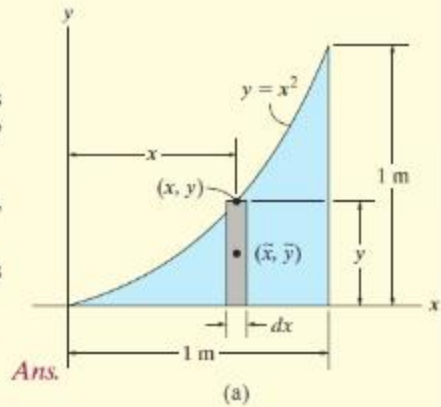


Fig. 9-12

Ex (5):

Determine the location of the center of mass of the cylinder shown in Fig. 9-15 if its density varies directly with the distance from its base, i.e.,  $\rho = 200z \text{ kg/m}^3$ .

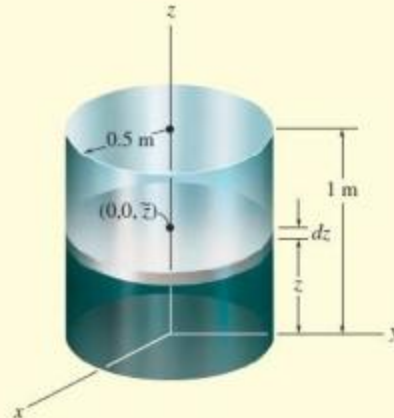


Fig. 9-15

**SOLUTION**

For reasons of material symmetry,

$$\bar{x} = \bar{y} = 0 \quad \text{Ans.}$$

**Differential Element.** A disk element of radius 0.5 m and thickness  $dz$  is chosen for integration, Fig. 9-15, since the *density of the entire element is constant* for a given value of  $z$ . The element is located along the  $z$  axis at the *arbitrary point*  $(0, 0, z)$ .

**Volume and Moment Arm.** The volume of the element is  $dV = \pi(0.5)^2 dz$ , and its centroid is located at  $\bar{z} = z$ .

**Integrations.** Using an equation similar to the third of Eqs. 9-2 and integrating with respect to  $z$ , noting that  $\rho = 200z$ , we have

$$\begin{aligned} \bar{z} &= \frac{\int_V \bar{z} \rho dV}{\int_V \rho dV} = \frac{\int_0^{1\text{ m}} z(200z)[\pi(0.5)^2 dz]}{\int_0^{1\text{ m}} (200z)\pi(0.5)^2 dz} \\ &= \frac{\int_0^{1\text{ m}} z^2 dz}{\int_0^{1\text{ m}} z dz} = 0.667 \text{ m} \quad \text{Ans.} \end{aligned}$$

## 7. Moment of Inertia:

The computation of the moment of the loading distribution about an axis will involve a quantity called the moment of inertia of the area. For example, consider the plate in Fig. 10–1, which is subjected to a fluid pressure  $p$  varies linearly with depth, such that  $p=\gamma y$ , where  $\gamma$  is the specific weight of the fluid.

Thus, the force acting on the differential area  $dA$  of the plate is  $dF=p.dA=(\gamma y)dA$ .

The moment of force about the x-axis is therefore

$$dM=y.dF=\gamma y^2 dA,$$

Integrating  $dM$  over entire area of the plate yields

$$M=\gamma \int y^2 dA.$$

The integral  $\int y^2 dA$  is called the moment of inertia of the area  $I_x$  about the  $x$ -axis.

Integrals of this form often arise in formulas used in, mechanic, and structural mechanic. The engineer needs to be familiar with methods used for their computation.

**Moment of Inertia.** By definition, the moments of inertia of a differential area  $dA$  about the  $x$  and  $y$  axes are  $dI_x=y^2 dA$ , and  $dI_y=x^2 dA$  respectively, Fig. 10–2.

For the entire area  $A$  the moments of inertia are determined by integration; i.e.,

$$I_x = \int_A y^2 dA$$

$$I_y = \int_A x^2 dA$$

The moment of inertia for  $dA$  about the “pole”  $O$  or  $z$ -axis, Fig.10-2, referred to as polar moment of inertia.  $dJ_o=r^2.dA$

Where:  $r$  is the perpendicular distance from the pole ( $z$  axis) to the element  $dA$ . For the entire area the polar moment of inertia is:

$$J_o = \int_A r^2 dA = I_x + I_y$$

The relation between  $J_o$  and  $I_x, I_y$  is possible since  $r^2=x^2+y^2$ .

Note:  $J_o$  and  $I_x, I_y$  is always positive.

The unites of moment of inertia raised to the power 4, e.g.  $m^4, mm^4$  or  $ft^4, in^4$ .

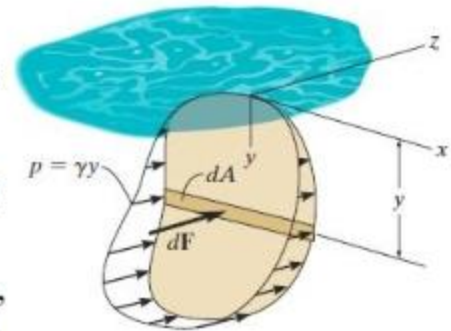


Fig. 10-1

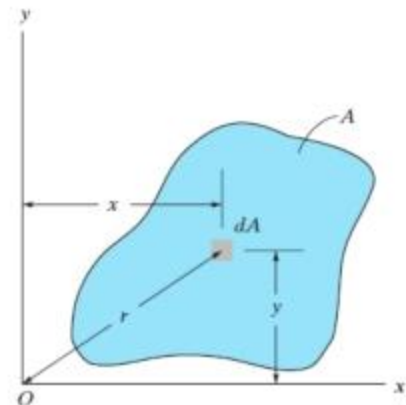


Fig. 10-2

### 7.1 Transfer Formula of the Moment of Inertia:

To find the moment of inertia of an area about any axis parallel to an axis passing through centroid where the moment of inertia is known.

To find the moment of inertia of the shaded area shown in Fig. 10–3 about the  $x$  axis, we choose a differential element  $dA$  located at distance  $y'$  from the centroidal  $x'$  axis. If the distance between the parallel  $x$  and  $x'$  axes is  $dy$  then the moment of inertia of  $dA$  about the  $x$  axis is

$$dI_x = (y' + dy)^2 dA .$$

For the entire area,

$$\begin{aligned} I_x &= \int_A (y' + d_y)^2 dA \\ &= \int_A y'^2 dA + 2d_y \int_A y' dA + d_y^2 \int_A dA \end{aligned}$$

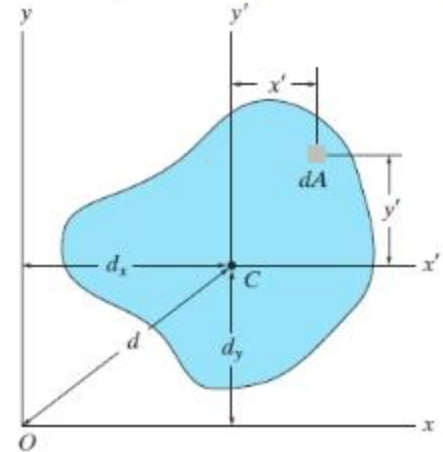


Fig. 10–3

The first integral is the moment of inertia about the centroidal axis  $I_{x'}$ .

The second integral is zero since the  $x'$  passes through area centroid  $C$ .

The third integral is the total area.

$$I_x = \bar{I}_{x'} + Ad_y^2 \quad (10-3)$$

A similar expression can be written for  $I_y$ ; i.e.,

$$I_y = \bar{I}_{y'} + Ad_x^2 \quad (10-4)$$

And finally, for the polar moment of inertia, since  $\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'}$  and  $d^2 = d_x^2 + d_y^2$ , we have

$$J_O = \bar{J}_C + Ad^2 \quad (10-5)$$

The form of each of these three equations states that *the moment of inertia for an area about an axis is equal to its moment of inertia about a parallel axis passing through the area's centroid plus the product of the area and the square of the perpendicular distance between the axes.*

## 7.2 Radius of Gyration of an Area:

The *radius of gyration* of an area about an axis has units of length and is a quantity that is often used for the design of columns in structural mechanics. Provided the areas and moments of inertia are *known*, the radii of gyration are determined from the formulas

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}} \quad (10-6)$$

$$k_O = \sqrt{\frac{J_O}{A}}$$

The form of these equations is easily remembered since it is similar to that for finding the moment of inertia for a differential area about an axis. For example,  $I_x = k_x^2 A$ ; whereas for a differential area,  $dI_x = y^2 dA$ .

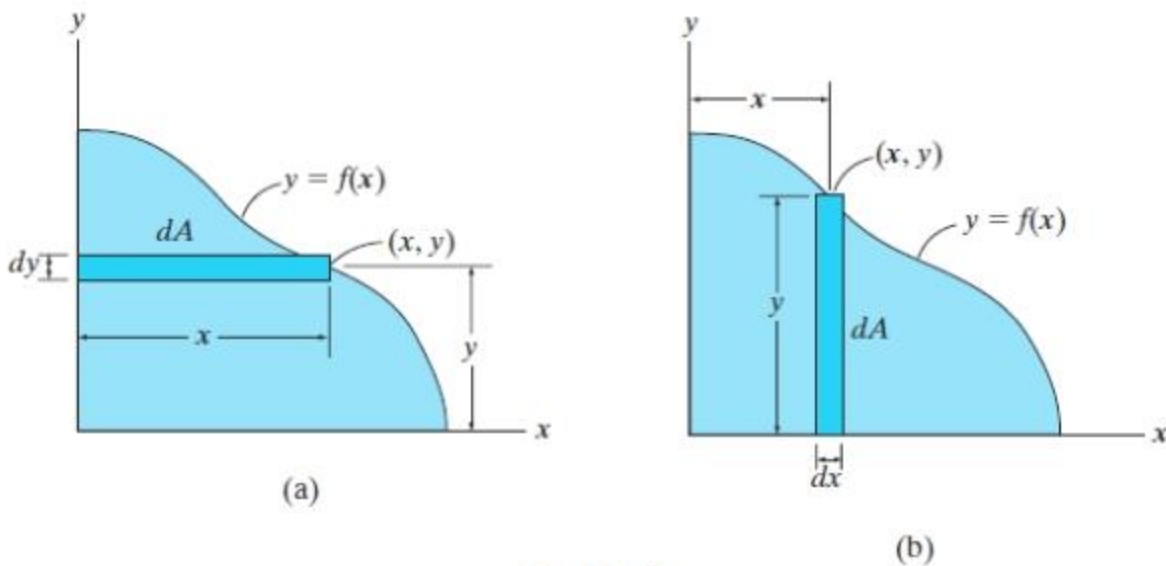


Fig. 10-4

Ex (1):

Determine the moment of inertia for the rectangular area shown in Fig. 10-5 with respect to (a) the centroidal  $x'$  axis, (b) the axis  $x_b$  passing through the base of the rectangle, and (c) the pole or  $z'$  axis perpendicular to the  $x'-y'$  plane and passing through the centroid  $C$ .

SOLUTION (CASE 1)

**Part (a).** The differential element shown in Fig. 10-5 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance  $y'$  from the  $x'$  axis. Here it is necessary to integrate from  $y' = -h/2$  to  $y' = h/2$ . Since  $dA = b dy'$ , then

$$\bar{I}_{x'} = \int_A y'^2 dA = \int_{-h/2}^{h/2} y'^2 (b dy') = b \int_{-h/2}^{h/2} y'^2 dy'$$

$$\bar{I}_{x'} = \frac{1}{12}bh^3 \quad \text{Ans.}$$

**Part (b).** The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the above result of part (a) and applying the parallel-axis theorem, Eq. 10-3.

$$\begin{aligned} I_{x_b} &= \bar{I}_{x'} + Ad_y^2 \\ &= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 \quad \text{Ans.} \end{aligned}$$

**Part (c).** To obtain the polar moment of inertia about point  $C$ , we must first obtain  $\bar{I}_{y'}$ , which may be found by interchanging the dimensions  $b$  and  $h$  in the result of part (a), i.e.,

$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

Using Eq. 10-2, the polar moment of inertia about  $C$  is therefore

$$J_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2) \quad \text{Ans.}$$

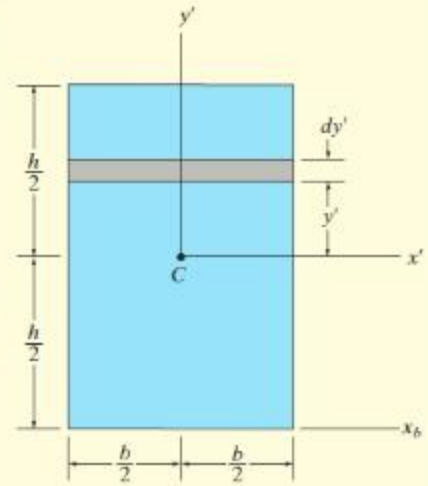
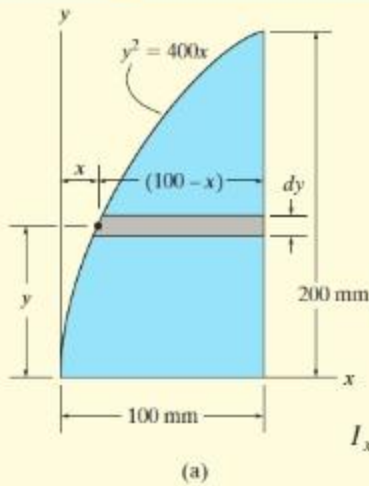


Fig. 10-5

Ex(2):



Determine the moment of inertia for the shaded area shown in Fig. 10-6a about the  $x$  axis.

#### SOLUTION I (CASE 1)

A differential element of area that is *parallel* to the  $x$  axis, as shown in Fig. 10-6a, is chosen for integration. Since this element has a thickness  $dy$  and intersects the curve at the *arbitrary point*  $(x, y)$ , its area is  $dA = (100 - x) dy$ . Furthermore, the element lies at the same distance  $y$  from the  $x$  axis. Hence, integrating with respect to  $y$ , from  $y = 0$  to  $y = 200$  mm, yields

$$\begin{aligned} I_x &= \int_A y^2 dA = \int_0^{200 \text{ mm}} y^2 (100 - x) dy \\ &= \int_0^{200 \text{ mm}} y^2 \left( 100 - \frac{y^2}{400} \right) dy = \int_0^{200 \text{ mm}} \left( 100y^2 - \frac{y^4}{400} \right) dy \\ &= 107(10^6) \text{ mm}^4 \end{aligned} \quad \text{Ans.}$$

#### SOLUTION II (CASE 2)

A differential element *parallel* to the  $y$  axis, as shown in Fig. 10-6b, is chosen for integration. It intersects the curve at the *arbitrary point*  $(x, y)$ . In this case, all points of the element do *not* lie at the same distance from the  $x$  axis, and therefore the parallel-axis theorem must be used to determine the *moment of inertia of the element* with respect to this axis. For a rectangle having a base  $b$  and height  $h$ , the moment of inertia about its centroidal axis has been determined in part (a) of Example 10.1. There it was found that  $\bar{I}_{x'} = \frac{1}{12}bh^3$ . For the differential element shown in Fig. 10-6b,  $b = dx$  and  $h = y$ , and thus  $d\bar{I}_{x'} = \frac{1}{12}dx y^3$ . Since the centroid of the element is  $\tilde{y} = y/2$  from the  $x$  axis, the moment of inertia of the element about this axis is

$$dI_x = d\bar{I}_{x'} + dA \tilde{y}^2 = \frac{1}{12}dx y^3 + y dx \left( \frac{y}{2} \right)^2 = \frac{1}{3}y^3 dx$$

(This result can also be concluded from part (b) of Example 10.1.) Integrating with respect to  $x$ , from  $x = 0$  to  $x = 100$  mm, yields

$$\begin{aligned} I_x &= \int dI_x = \int_0^{100 \text{ mm}} \frac{1}{3}y^3 dx = \int_0^{100 \text{ mm}} \frac{1}{3}(400x)^{3/2} dx \\ &= 107(10^6) \text{ mm}^4 \end{aligned} \quad \text{Ans.}$$

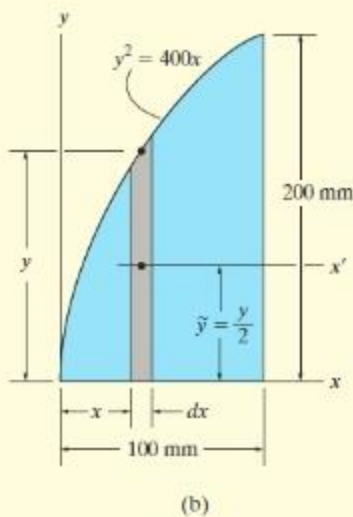
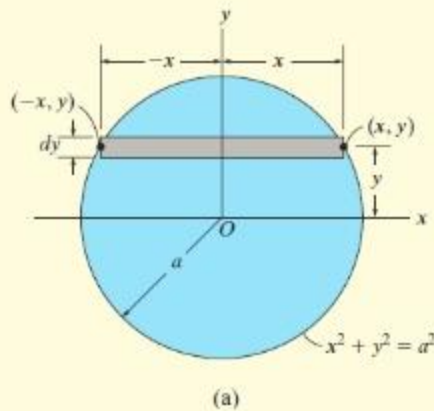


Fig. 10-6

Ex (3):

Determine the moment of inertia with respect to the  $x$  axis for the circular area shown in Fig. 10-7a.



**SOLUTION I (CASE 1)**

Using the differential element shown in Fig. 10-7a, since  $dA = 2x dy$ , we have

$$\begin{aligned}
 I_x &= \int_A y^2 dA = \int_A y^2 (2x) dy \\
 &= \int_{-a}^a y^2 (2\sqrt{a^2 - y^2}) dy = \frac{\pi a^4}{4} \quad \text{Ans.}
 \end{aligned}$$

**SOLUTION II (CASE 2)**

When the differential element shown in Fig. 10-7b is chosen, the centroid for the element happens to lie on the  $x$  axis, and since  $\bar{I}_{x'} = \frac{1}{12}bh^3$  for a rectangle, we have

$$\begin{aligned}
 dI_x &= \frac{1}{12} dx (2y)^3 \\
 &= \frac{2}{3} y^3 dx
 \end{aligned}$$

Integrating with respect to  $x$  yields

$$I_x = \int_{-a}^a \frac{2}{3} (a^2 - x^2)^{3/2} dx = \frac{\pi a^4}{4} \quad \text{Ans.}$$

**NOTE:** By comparison, Solution I requires much less computation. Therefore, if an integral using a particular element appears difficult to evaluate, try solving the problem using an element oriented in the other direction.

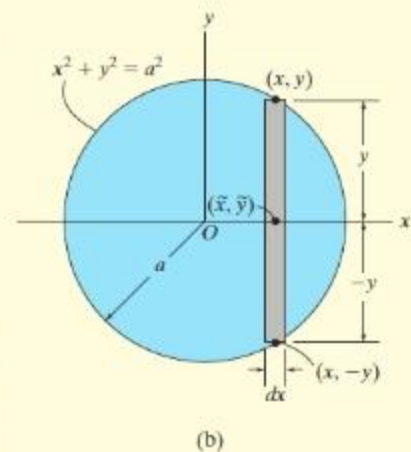


Fig. 10-7

### 7.3 Moment of Inertia for Composite Area:

A composite area consists of a series of connected “simpler” parts or shapes, such as rectangles, triangles, and circles. Provided the moment of inertia of each of these parts is known or can be determined about a common axis.

The moment of inertia for the composite area about this axis equals the algebraic sum of the moments of inertia of all its parts.

#### Ex (1):

Determine the moment of inertia of the area shown in Fig. 10–8a about the  $x$  axis.

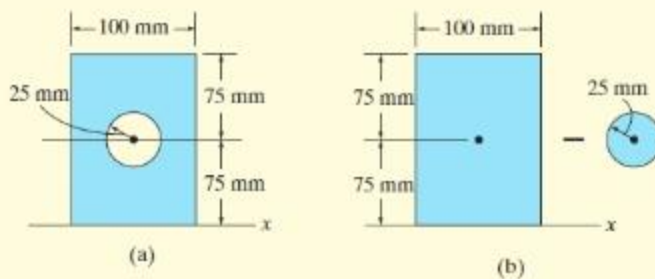


Fig. 10–8

#### SOLUTION

**Composite Parts.** The area can be obtained by *subtracting* the circle from the rectangle shown in Fig. 10–8b. The centroid of each area is located in the figure.

**Parallel-Axis Theorem.** The moments of inertia about the  $x$  axis are determined using the parallel-axis theorem and the data in the table on the inside back cover.

*Circle*

$$\begin{aligned} I_x &= \bar{I}_{x'} + Ad_y^2 \\ &= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6) \text{ mm}^4 \end{aligned}$$

*Rectangle*

$$\begin{aligned} I_x &= \bar{I}_{x'} + Ad_y^2 \\ &= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6) \text{ mm}^4 \end{aligned}$$

**Summation.** The moment of inertia for the area is therefore

$$\begin{aligned} I_x &= -11.4(10^6) + 112.5(10^6) \\ &= 101(10^6) \text{ mm}^4 \end{aligned} \quad \text{Ans.}$$

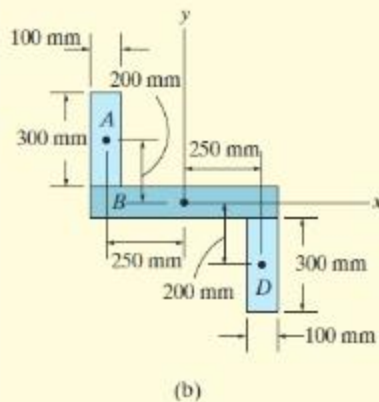
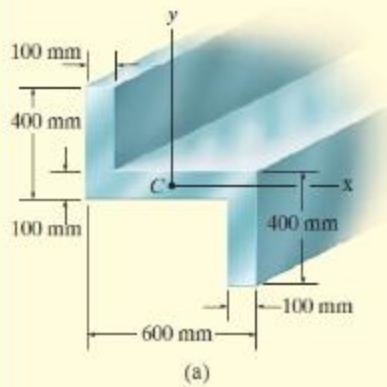
**Ex (2):**

Fig. 10-9

Determine the moments of inertia for the cross-sectional area of the member shown in Fig. 10-9a about the  $x$  and  $y$  centroidal axes.

**SOLUTION**

**Composite Parts.** The cross section can be subdivided into the three rectangular areas  $A$ ,  $B$ , and  $D$  shown in Fig. 10-9b. For the calculation, the centroid of each of these rectangles is located in the figure.

**Parallel-Axis Theorem.** From the table on the inside back cover, or Example 10.1, the moment of inertia of a rectangle about its centroidal axis is  $\bar{I} = \frac{1}{12}bh^3$ . Hence, using the parallel-axis theorem for rectangles  $A$  and  $D$ , the calculations are as follows:

**Rectangles A and D**

$$I_x = \bar{I}_x + Ad_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2 = 1.425(10^9) \text{ mm}^4$$

$$I_y = \bar{I}_y + Ad_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2 = 1.90(10^9) \text{ mm}^4$$

**Rectangle B**

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9) \text{ mm}^4$$

$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9) \text{ mm}^4$$

**Summation.** The moments of inertia for the entire cross section are thus

$$I_x = 2[1.425(10^9)] + 0.05(10^9) = 2.90(10^9) \text{ mm}^4 \quad \text{Ans.}$$

$$I_y = 2[1.90(10^9)] + 1.80(10^9) = 5.60(10^9) \text{ mm}^4 \quad \text{Ans.}$$

## 7.4 Product of Inertia for an Area:

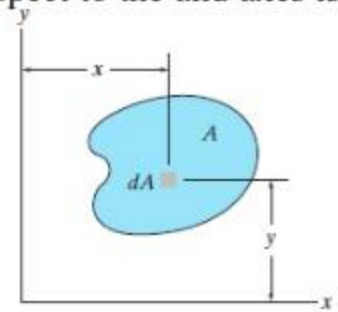
The product of inertia, is required in order to determine the maximum and minimum moments of inertia for the area. These maximum and minimum values are important properties needed for designing structural and mechanical members such as beams, columns, and shafts.

The product of inertia of the area in Fig. 10–10 with respect to the  $x$  and  $y$  axes is defined as

$$I_{xy} = \int_A xy \, dA$$

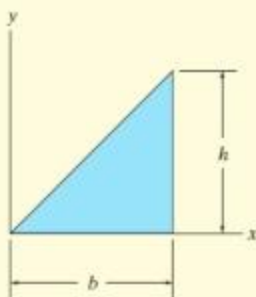
As the moment of inertia the transfer of the product of inertia can be estimated as:

$$I_{xy} = \bar{I}_{x'y'} + Ad_xd_y$$



Ex(1):

Fig. 10–10



(a)

Determine the product of inertia  $I_{xy}$  for the triangle shown in Fig. 10–14a.

### SOLUTION I

A differential element that has a thickness  $dx$ , as shown in Fig. 10–14b, has an area  $dA = y \, dx$ . The product of inertia of this element with respect to the  $x$  and  $y$  axes is determined using the parallel-axis theorem.

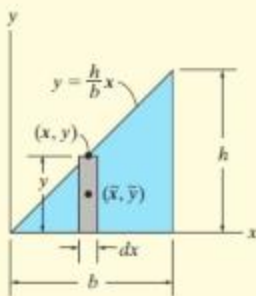
$$dI_{xy} = d\bar{I}_{x'y'} + dA \tilde{x} \tilde{y}$$

where  $\tilde{x}$  and  $\tilde{y}$  locate the *centroid* of the element or the origin of the  $x'$ ,  $y'$  axes. (See Fig. 10–13.) Since  $d\bar{I}_{x'y'} = 0$ , due to symmetry, and  $\tilde{x} = x$ ,  $\tilde{y} = y/2$ , then

$$\begin{aligned} dI_{xy} &= 0 + (y \, dx)x\left(\frac{y}{2}\right) = \left(\frac{h}{b}x \, dx\right)x\left(\frac{h}{2b}x\right) \\ &= \frac{h^2}{2b^2}x^3 \, dx \end{aligned}$$

Integrating with respect to  $x$  from  $x = 0$  to  $x = b$  yields

$$I_{xy} = \frac{h^2}{2b^2} \int_0^b x^3 \, dx = \frac{b^2 h^2}{8} \quad \text{Ans.}$$



(b)

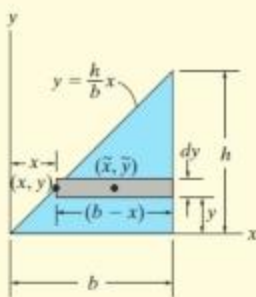
### SOLUTION II

The differential element that has a thickness  $dy$ , as shown in Fig. 10–14c, can also be used. Its area is  $dA = (b - x) \, dy$ . The *centroid* is located at point  $\tilde{x} = x + (b - x)/2 = (b + x)/2$ ,  $\tilde{y} = y$ , so the product of inertia of the element becomes

$$\begin{aligned} dI_{xy} &= d\bar{I}_{x'y'} + dA \tilde{x} \tilde{y} \\ &= 0 + (b - x) \, dy \left(\frac{b + x}{2}\right) y \\ &= \left(b - \frac{b}{h}y\right) dy \left[\frac{b + (b/h)y}{2}\right] y = \frac{1}{2}y \left(b^2 - \frac{b^2}{h^2}y^2\right) dy \end{aligned}$$

Integrating with respect to  $y$  from  $y = 0$  to  $y = h$  yields

$$I_{xy} = \frac{1}{2} \int_0^h y \left(b^2 - \frac{b^2}{h^2}y^2\right) dy = \frac{b^2 h^2}{8} \quad \text{Ans.}$$



(c)

Fig. 10–14

Ex (2):

Determine the product of inertia for the cross-sectional area of the member shown in Fig. 10-15a, about the  $x$  and  $y$  centroidal axes.

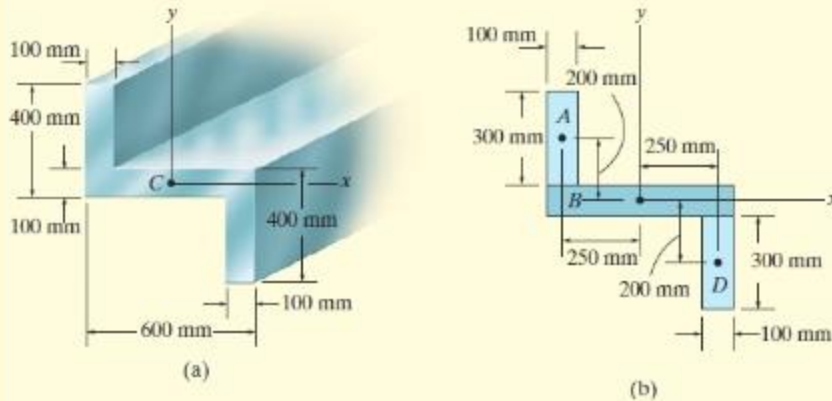


Fig. 10-15

**SOLUTION**

As in Example 10.5, the cross section can be subdivided into three composite rectangular areas  $A$ ,  $B$ , and  $D$ , Fig. 10-15b. The coordinates for the centroid of each of these rectangles are shown in the figure. Due to symmetry, the product of inertia of *each rectangle is zero* about a set of  $x'$ ,  $y'$  axes that passes through the centroid of each rectangle. Using the parallel-axis theorem, we have

*Rectangle A*

$$I_{xy} = \bar{I}_{x'y'} + Ad_xd_y$$

$$= 0 + (300)(100)(-250)(200) = -1.50(10^9) \text{ mm}^4$$

*Rectangle B*

$$I_{xy} = \bar{I}_{x'y'} + Ad_xd_y$$

$$= 0 + 0 = 0$$

*Rectangle D*

$$I_{xy} = \bar{I}_{x'y'} + Ad_xd_y$$

$$= 0 + (300)(100)(250)(-200) = -1.50(10^9) \text{ mm}^4$$

The product of inertia for the entire cross section is therefore

$$I_{xy} = -1.50(10^9) + 0 - 1.50(10^9) = -3.00(10^9) \text{ mm}^4 \text{ Ans.}$$

**NOTE:** This negative result is due to the fact that rectangles  $A$  and  $D$  have centroids located with negative  $x$  and negative  $y$  coordinates, respectively.

## 7.5 Moments of Inertia for an Area about Inclined Axes:

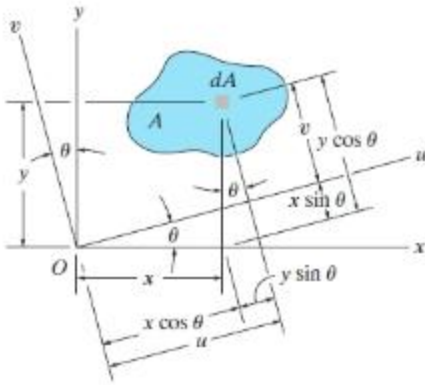


Fig. 10-16

In structural and mechanical design, it is sometimes necessary to calculate the moments and product of inertia  $I_u$ ,  $I_v$ , and  $I_{uv}$  for an area with respect to a set of inclined  $u$  and  $v$  axes when the values for  $\theta$ ,  $I_x$ ,  $I_y$ , and  $I_{xy}$  are known. To do this we will use *transformation equations* which relate the  $x$ ,  $y$  and  $u$ ,  $v$  coordinates. From Fig. 10-16, these equations are

$$u = x \cos \theta + y \sin \theta$$

$$v = y \cos \theta - x \sin \theta$$

With these equations, the moments and product of inertia of  $dA$  about the  $u$  and  $v$  axes become

$$dI_u = v^2 dA = (y \cos \theta - x \sin \theta)^2 dA$$

$$dI_v = u^2 dA = (x \cos \theta + y \sin \theta)^2 dA$$

$$dI_{uv} = uv dA = (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta) dA$$

Expanding each expression and integrating, realizing that  $I_x = \int y^2 dA$ ,  $I_y = \int x^2 dA$ , and  $I_{xy} = \int xy dA$ , we obtain

$$I_u = I_x \cos^2 \theta + I_y \sin^2 \theta - 2I_{xy} \sin \theta \cos \theta$$

$$I_v = I_x \sin^2 \theta + I_y \cos^2 \theta + 2I_{xy} \sin \theta \cos \theta$$

$$I_{uv} = I_x \sin \theta \cos \theta - I_y \sin \theta \cos \theta + I_{xy}(\cos^2 \theta - \sin^2 \theta)$$

Using the trigonometric identities  $\sin 2\theta = 2 \sin \theta \cos \theta$  and  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$  we can simplify the above expressions, in which case

$$\begin{aligned} I_u &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \\ I_v &= \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \\ I_{uv} &= \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \end{aligned} \quad (10-9)$$

Notice that if the first and second equations are added together, we can show that the polar moment of inertia about the  $z$  axis passing through point  $O$  is, as expected, *independent* of the orientation of the  $u$  and  $v$  axes; i.e.,

$$J_O = I_u + I_v = I_x + I_y$$

**Principal Moments of Inertia.** Equations 10-9 show that  $I_u$ ,  $I_v$ , and  $I_{uv}$  depend on the angle of inclination,  $\theta$ , of the  $u$ ,  $v$  axes. We will now determine the orientation of these axes about which the moments of inertia for the area are maximum and minimum. This particular set of axes is called the *principal axes* of the area, and the corresponding moments of inertia with respect to these axes are called the *principal moments of inertia*. In general, there is a set of principal axes for every chosen origin  $O$ . However, for structural and mechanical design, the origin  $O$  is located at the centroid of the area.

The angle which defines the orientation of the principal axes can be found by differentiating the first of Eqs. 10-9 with respect to  $\theta$  and setting the result equal to zero. Thus,

$$\frac{dI_u}{d\theta} = -2\left(\frac{I_x - I_y}{2}\right) \sin 2\theta - 2I_{xy} \cos 2\theta = 0$$

Therefore, at  $\theta = \theta_p$ ,

$$\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2} \quad (10-10)$$

The two roots  $\theta_{p_1}$  and  $\theta_{p_2}$  of this equation are  $90^\circ$  apart, and so they each specify the inclination of one of the principal axes. In order to substitute them into Eq. 10-9, we must first find the sine and cosine of  $2\theta_{p_1}$  and  $2\theta_{p_2}$ . This can be done using these ratios from the triangles shown in Fig. 10-17, which are based on Eq. 10-10.

Substituting each of the sine and cosine ratios into the first or second of Eqs. 10-9 and simplifying, we obtain

$$I_{\max} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \quad (10-11)$$

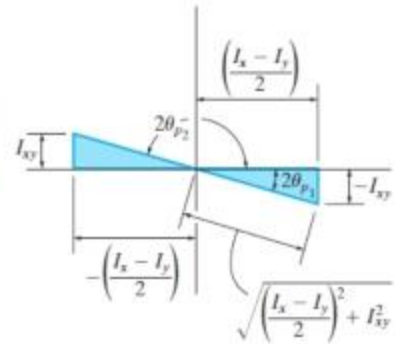
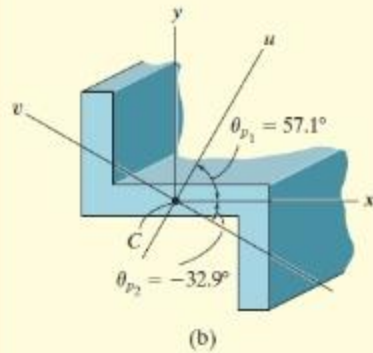
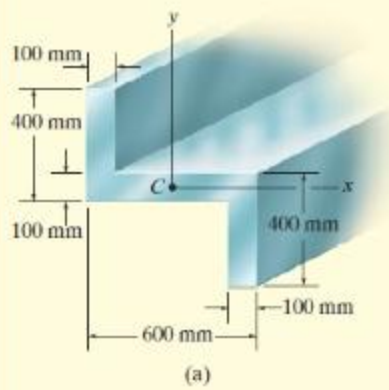


Fig. 10-17

Depending on the sign chosen, this result gives the maximum or minimum moment of inertia for the area. Furthermore, if the above trigonometric relations for  $\theta_{p_1}$  and  $\theta_{p_2}$  are substituted into the third of Eqs. 10-9, it can be shown that  $I_{uv} = 0$ ; that is, the *product of inertia with respect to the principal axes is zero*. Since it was indicated in Sec. 10.6 that the product of inertia is zero with respect to any symmetrical axis, it therefore follows that *any symmetrical axis represents a principal axis of inertia for the area*.

**EXAMPLE 10.8**

**Fig. 10-18**

Determine the principal moments of inertia and the orientation of the principal axes for the cross-sectional area of the member shown in Fig. 10-18a with respect to an axis passing through the centroid.

**SOLUTION**

The moments and product of inertia of the cross section with respect to the  $x, y$  axes have been determined in Examples 10.5 and 10.7. The results are

$$I_x = 2.90(10^9) \text{ mm}^4 \quad I_y = 5.60(10^9) \text{ mm}^4 \quad I_{xy} = -3.00(10^9) \text{ mm}^4$$

Using Eq. 10-10, the angles of inclination of the principal axes  $u$  and  $v$  are

$$\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2} = \frac{-[-3.00(10^9)]}{[2.90(10^9) - 5.60(10^9)]/2} = -2.22$$

$$2\theta_p = -65.8^\circ \text{ and } 114.2^\circ$$

Thus, by inspection of Fig. 10-18b,

$$\theta_{p_2} = -32.9^\circ \quad \text{and} \quad \theta_{p_1} = 57.1^\circ \quad \text{Ans.}$$

The principal moments of inertia with respect to these axes are determined from Eq. 10-11. Hence,

$$I_{\max} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{2.90(10^9) + 5.60(10^9)}{2}$$

$$\pm \sqrt{\left[\frac{2.90(10^9) - 5.60(10^9)}{2}\right]^2 + [-3.00(10^9)]^2}$$

$$I_{\max} = 4.25(10^9) \pm 3.29(10^9)$$

or

$$I_{\max} = 7.54(10^9) \text{ mm}^4 \quad I_{\min} = 0.960(10^9) \text{ mm}^4 \quad \text{Ans.}$$

**NOTE:** The maximum moment of inertia,  $I_{\max} = 7.54(10^9) \text{ mm}^4$ , occurs with respect to the  $u$  axis since *by inspection* most of the cross-sectional area is farthest away from this axis. Or, stated in another manner,  $I_{\max}$  occurs about the  $u$  axis since this axis is located within  $\pm 45^\circ$  of the  $y$  axis, which has the larger value of  $I$  ( $I_y > I_x$ ). Also, this can be concluded by substituting the data with  $\theta = 57.1^\circ$  into the first of Eqs. 10-9 and solving for  $I_u$ .